

# STABILITY CONDITIONS AND CURVE COUNTING INVARIANTS ON CALABI-YAU 3-FOLDS

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**ABSTRACT.** The purpose of this paper is twofold: first we give a survey on the recent developments of curve counting invariants on Calabi-Yau 3-folds, e.g. Gromov-Witten theory, Donaldson-Thomas theory and Pandharipande-Thomas theory. Next we focus on the proof of the rationality conjecture of the generating series of PT invariants, and discuss its conjectural Gopakumar-Vafa form.

## 1. INTRODUCTION

1.1. **Background.** Let  $X$  be a smooth projective Calabi-Yau 3-fold, i.e.

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

We are interested in the curve counting theory on  $X$ . This is an important field of study in connection with mirror symmetry: it predicts a relationship between curve counting invariants on  $X$  and a period integral on its mirror manifold  $\check{X}$ . So far curve counting invariants have been computed and compared under the mirror symmetry in several situations.

Now there are three kinds of curve counting theories on  $X$ :

- **Gromov-Witten (GW) theory:** counting pairs,

$$(C, f), \quad f: C \rightarrow X,$$

where  $C$  is a connected nodal curve and  $f$  is a morphism with finite automorphisms. In terms of string theory, GW invariants count *world sheets*. The moduli space defining the GW theory is Kontsevich's stable map moduli space. The resulting invariants are  $\mathbb{Q}$ -valued.

- **Donaldson-Thomas (DT) theory:** counting subschemes,

$$Z \subset X,$$

with  $\dim Z \leq 1$ . In terms of string theory, DT invariants count *D-branes*. The moduli space defining the DT theory is the classical Hilbert scheme. The resulting invariants are  $\mathbb{Z}$ -valued.

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- **Pandharipande-Thomas (PT) theory:** counting pairs,

$$(F, s), \quad s: \mathcal{O}_X \rightarrow F,$$

where  $F$  is a pure one dimensional sheaf, and  $s$  is surjective in dimension one. The PT invariants also count D-branes, but the stability condition is different from DT theory. The moduli space defining the PT theory is identified with the moduli space of two term complexes,

$$I^\bullet = (\mathcal{O}_X \xrightarrow{s} F) \in D^b \text{Coh}(X).$$

Here  $D^b \text{Coh}(X)$  is the bounded derived category of coherent sheaves on  $X$ .

An equivalence between GW and DT theories is conjectured by Maulik-Nekrasov-Okounkov-Pandharipande [30]. Also an equivalence between DT and PT theories is conjectured by Pandharipande-Thomas [32]. They are formulated in terms of generating functions.

On the other hand, the notion of stability conditions on  $D^b \text{Coh}(X)$  is introduced by Bridgeland [10]. He shows that the set of stability conditions on  $D^b \text{Coh}(X)$ , denoted by

$$\text{Stab}(X),$$

has a structure of a complex manifold. The space  $\text{Stab}(X)$  is expected to be related to the stringy Kähler moduli space, which should be isomorphic to the moduli space of complex structures of the mirror  $\check{X}$ . An important observation by Pandharipande-Thomas [32] is that the DT/PT correspondence should be interpreted as wall-crossing phenomena in the space of stability conditions  $\text{Stab}(X)$ . Although it is still difficult to study  $\text{Stab}(X)$  when  $X$  is a projective Calabi-Yau 3-fold, kinds of ‘limiting degenerations’ of Bridgeland stability have been introduced in [1], [36], [37], and DT/PT wall-crossing is also observed in these degenerated stability conditions.

In recent years, the wall-crossing formula of DT type invariants have been established by Joyce-Song [19] and Kontsevich-Soibelman [23] in a general setting. Since then, it turns out that a categorical approach is useful in the study of DT type curve counting invariants. Now several applications have been obtained, e.g. DT/PT correspondence, rationality conjecture. (cf. [9], [33], [37], [38].) One of the purposes of this paper is to give a survey of these recent developments.

As for another purpose, we focus on the rationality conjecture of the generating series of PT invariants proposed in [32]. The Euler characteristic version is proved in [38], and the virtual cycle is involved in [9]. In this paper, assuming the announced result by Behrend-Getzler [6], we give its another proof by discussing in the framework of [37]. The main idea is the same as in [38], but the argument is simplified. We also discuss a conjectural Gopakumar-Vafa form of the generating series of

PT invariants, and see that it is related to the multi-covering formula of generalized DT invariants introduced by Joyce-Song [19]. We also give an evidence of the conjectural multi-covering formula when  $X$  is a certain elliptically fibered Calabi-Yau 3-fold.

**1.2. Plan of the paper.** In Section 2, we give a survey on stability conditions. In Section 3, we recall several curve counting invariants on Calabi-Yau 3-folds and the relevant conjectures, results. In Section 4, we recall the notion of Hall algebras and the generalized DT invariants counting one dimensional sheaves. In Section 5, we give a proof of the rationality of the generating series of PT invariants in the framework of [37]. In Section 6, we discuss a Gopakumar-Vafa form of the generating series of PT invariants, and the multi-covering formula of generalized DT invariants.

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**1.4. Notation and Convention.** For a triangulated category  $\mathcal{D}$ , the shift functor is denoted by  $[1]$ . For a set of objects  $\mathcal{S} \subset \mathcal{D}$ , we denote by  $\langle \mathcal{S} \rangle_{\text{tr}}$  the smallest triangulated subcategory which contains  $\mathcal{S}$  and  $0 \in \mathcal{D}$ . Also we denote by  $\langle \mathcal{S} \rangle_{\text{ex}}$  the smallest extension closed subcategory of  $\mathcal{D}$  which contains  $\mathcal{S}$  and  $0 \in \mathcal{D}$ . The abelian category of coherent sheaves on a variety  $X$  is denoted by  $\text{Coh}(X)$ . We say  $F \in \text{Coh}(X)$  is  $d$ -dimensional if its support is  $d$ -dimensional. We always assume that the second homology group  $H_2(X, \mathbb{Z})$  is torsion free. If there is a torsion, then the arguments are applied if we replace  $H_2(X, \mathbb{Z})$  by its torsion free part. For  $\beta \in H_2(X, \mathbb{Z})$ , we write  $\beta > 0$  if  $\beta$  is a class of an effective algebraic one cycle on  $X$ .

## 2. STABILITY CONDITIONS

We begin with recalling stability conditions on abelian categories, and explain typical wall-crossing phenomena.

**2.1. Definitions of stability conditions.** Classically there is a notion of a stability condition on vector bundles on smooth projective curves. Let  $C$  be a smooth projective curve over  $\mathbb{C}$  and  $E$  a vector bundle on it. The slope of  $E$  is defined by

$$\mu(E) := \deg(E)/\text{rank}(E).$$

**Definition 2.1.** A vector bundle  $E$  on  $C$  is (semi)stable if for any subbundle  $0 \neq F \subsetneq E$ , we have

$$\mu(F) < (\leq) \mu(E).$$

We have the following properties:

- If we fix rank  $r$  and degree  $d$ , then there is a good moduli space of slope semistable vector bundles  $E$  with  $\text{rank}(E) = r$  and  $\deg(E) = d$ .
- For any vector bundle  $E$  on  $C$ , there is a filtration, (Harder-Narasimhan filtration,)

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E,$$

such that each subquotient  $F_i = E_i/E_{i-1}$  is semistable with  $\mu(F_i) > \mu(F_{i+1})$  for all  $i$ .

A stability condition on an abelian category is defined to be a direct generalization of the above classical notion. Let  $\mathcal{A}$  be an abelian category, e.g. the category of coherent sheaves on an algebraic variety. Recall that its Grothendieck group is defined by

$$K(\mathcal{A}) := \bigoplus_{E \in \mathcal{A}} \mathbb{Z}[E] / \sim,$$

where the equivalence relation  $\sim$  is generated by

$$[E_2] \sim [E_1] + [E_3],$$

for all exact sequences  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  in  $\mathcal{A}$ . We fix a finitely generated abelian group  $\Gamma$  together with a group homomorphism,

$$\text{cl}: K(\mathcal{A}) \rightarrow \Gamma.$$

For instance if  $\mathcal{A} = \text{Coh}(X)$  for a smooth projective variety  $X$ , we can take  $\Gamma$  to be the image of the Chern character map,

$$(1) \quad \text{ch}: K(\mathcal{A}) \twoheadrightarrow \Gamma \subset H^*(X, \mathbb{Q}),$$

and  $\text{cl} = \text{ch}$ . Let  $\mathbb{H} \subset \mathbb{C}$  be the subset

$$\mathbb{H} = \{r \exp(\pi i \phi) : r > 0, 0 < \phi \leq 1\}.$$

The following formulation of stability conditions is due to Bridgeland [10].

**Definition 2.2.** A stability condition on  $\mathcal{A}$  is a group homomorphism,

$$Z: \Gamma \rightarrow \mathbb{C},$$

satisfying the following axiom.

(i) For any non-zero object  $E \in \mathcal{A}$ , we have

$$Z(E) := Z(\text{cl}(E)) \in \mathbb{H}.$$

In particular the argument

$$\arg Z(E) \in (0, \pi],$$

is well-defined. An object  $E \in \mathcal{A}$  is called  $Z$ -(semi)stable if for any non-zero subobject  $0 \neq F \subsetneq E$ , we have

$$\arg Z(F) < (\leq) \arg Z(E).$$

(ii) For any object  $E \in \mathcal{A}$ , there is a filtration, (Harder-Narasimhan filtration,)

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E,$$

such that each subquotient  $F_i = E_i/E_{i-1}$  is  $Z$ -semistable with

$$\arg Z(F_1) > \arg Z(F_2) > \cdots > \arg Z(F_N).$$

Here we give some examples.

**Example 2.3.** (i) Let  $C$  be a smooth projective curve over  $\mathbb{C}$  and take  $\mathcal{A} = \text{Coh}(C)$ . We set  $\Gamma$  to be

$$\Gamma = \mathbb{Z} \oplus \mathbb{Z},$$

and a group homomorphism  $\text{cl}: K(C) \rightarrow \Gamma$  to be

$$\text{cl}(E) = (\text{rank}(E), \deg(E)).$$

Let  $Z: \Gamma \rightarrow \mathbb{C}$  be the map defined by

$$Z(r, d) = -d + \sqrt{-1}r.$$

Then it is easy to see that  $Z$  is a stability condition on  $\text{Coh}(C)$ . An object  $E \in \text{Coh}(C)$  is  $Z$ -semistable if and only if  $E$  is a torsion sheaf or  $E$  is a semistable vector bundle in the sense of Definition 2.1.

(ii) Let  $A$  be a finite dimensional algebra over  $\mathbb{C}$  and  $\mathcal{A}$  the abelian category of finitely generated right  $A$ -modules. There is a finite number of simple objects  $S_1, S_2, \dots, S_N$  in  $\mathcal{A}$  such that

$$K(\mathcal{A}) \cong \bigoplus_{i=1}^N \mathbb{Z}[S_i].$$

We set  $\Gamma = K(\mathcal{A})$  and  $\text{cl} = \text{id}$ . Choose elements,

$$z_1, z_2, \dots, z_N \in \mathbb{H}.$$

Then the map  $Z: \Gamma \rightarrow \mathbb{C}$  defined by

$$Z\left(\sum_i a_i [S_i]\right) = \sum_i a_i z_i,$$

is a stability condition on  $\mathcal{A}$ .

(iii) The following generalization of (i) will be used in the later sections. Let  $X$  be a smooth projective variety over  $\mathbb{C}$ . We set

$$\text{Coh}_{\leq 1}(X) := \{E \in \text{Coh}(X) : \dim \text{Supp}(E) \leq 1\}.$$

We set

$$\Gamma_0 := \mathbb{Z} \oplus H_2(X, \mathbb{Z}),$$

and the group homomorphism  $\text{cl}_0: K(\text{Coh}_{\leq 1}(X)) \rightarrow \Gamma_0$  to be

$$\text{cl}_0(E) := (\text{ch}_3(E), \text{ch}_2(E)).$$

By the Riemann-Roch theorem,  $\text{cl}_0(E)$  is also written as  $(\chi(E), [E])$ , where  $[E]$  is the fundamental homology class determined by  $E$  and  $\chi(E)$  is the holomorphic Euler characteristic.

Let  $\omega$  be an  $\mathbb{R}$ -ample divisor on  $X$ . We set  $Z_\omega: \Gamma_0 \rightarrow \mathbb{C}$  to be

$$Z_\omega(n, \beta) := -n + (\omega \cdot \beta)\sqrt{-1}.$$

Then  $Z_\omega$  is a stability condition on  $\text{Coh}_{\leq 1}(X)$ . An object  $E \in \text{Coh}_{\leq 1}(X)$  is  $Z_\omega$ -(semi)stable iff  $E$  is  $\omega$ -Gieseker (semi)stable sheaf. (cf. [16].) If  $\dim X = 1$  and  $\deg \omega = 1$ , then  $Z_\omega$  coincides with the stability condition constructed in (i).

**2.2. Wall-crossing phenomena.** Here we explain a rough idea of wall-crossing phenomena and a simple example. We set

$$\text{Stab}(\mathcal{A}) := \{Z \in \Gamma_{\mathbb{C}}^\vee : Z \text{ is a stability condition on } \mathcal{A}\}.$$

For instance in Example 2.3 (ii), we have the identification,

$$\text{Stab}(\mathcal{A}) \cong \mathbb{H}^N.$$

For  $v \in \Gamma$ , we are interested in ‘counting invariants’,

$$\text{Stab}(\mathcal{A}) \ni Z \mapsto I_v(Z) \in \mathbb{Q},$$

where  $I_v(Z)$  ‘counts’  $Z$ -semistable objects  $E \in \mathcal{A}$  with  $\text{cl}(E) = v$ . There may be several choices of the definition of  $I_v(Z)$ . For instance we can consider moduli space of  $Z$ -semistable objects  $E \in \mathcal{A}$  with  $\text{cl}(E) = v$ , denoted by  $M_v(Z)$ , and take  $I_v(Z)$  to be

$$I_v(Z) = \chi(M_v(Z)).$$

Here  $\chi(*)$  is the topological Euler characteristic. We need to check that the existence of the moduli space  $M_v(Z)$ , but this holds in the cases given in Example 2.3.

In principle, there should be a wall and chamber structure on the space  $\text{Stab}(\mathcal{A})$  such that  $I_v(Z)$  is constant on a chamber but jumps on a wall. The set of walls is given by a countable number of real codimension one submanifolds  $\{W_\lambda\}_{\lambda \in \Lambda}$  in  $\text{Stab}(\mathcal{A})$ , and a chamber is a connected component,

$$\mathcal{C} \subset \text{Stab}(\mathcal{A}) \setminus \bigcup_{\lambda \in \Lambda} W_\lambda.$$

For instance, let us consider the algebra  $A$  given by

$$A = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}.$$

Let  $\mathcal{A}$  be the abelian category of finitely generated right  $A$ -modules. (In other words,  $\mathcal{A}$  is the category of representations of a quiver with two vertex and one arrow.) There are two simple objects in  $\mathcal{A}$ ,

$$S_i = \mathbb{C} \cdot e_i, \quad i = 1, 2,$$

whose right  $A$ -actions are given by

$$e_i \cdot \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix} = a_i e_i.$$

We take an object  $E \in \mathcal{A}$ , which is isomorphic to  $\mathbb{C}^2$  as a  $\mathbb{C}$ -vector space, and the right  $A$ -action is the standard one. There is an exact sequence in  $\mathcal{A}$ ,

$$(2) \quad 0 \rightarrow S_2 \rightarrow E \rightarrow S_1 \rightarrow 0.$$

Let us identify  $\text{Stab}(\mathcal{A})$  with  $\mathbb{H}^2$ , as in Example 2.3 (ii). For a stability condition

$$Z = (z_1, z_2) \in \text{Stab}(\mathcal{A}) \cong \mathbb{H}^2,$$

the exact sequence (2) easily implies the following.

$$E \text{ is } \begin{cases} Z\text{-stable} & \text{if } \arg z_2 < \arg z_1 \\ Z\text{-semistable} & \text{if } \arg z_2 = \arg z_1 \\ \text{not } Z\text{-semistable} & \text{if } \arg z_2 > \arg z_1 \end{cases}$$

In particular for an element

$$v = \text{cl}(E) = (1, 1) \in \Gamma,$$

the moduli space  $M_v(Z)$  is

$$M_v(Z) = \begin{cases} \{E\} & \text{if } \arg z_2 < \arg z_1 \\ \{E\} \cup \{S_1 \oplus S_2\} & \text{if } \arg z_2 = \arg z_1 \\ \emptyset & \text{if } \arg z_2 > \arg z_1 \end{cases}$$

The ‘counting invariant’  $I_v(Z) = \chi(M_v(Z))$  is

$$I_v(Z) = \begin{cases} 1 & \text{if } \arg z_2 < \arg z_1 \\ 2 & \text{if } \arg z_2 = \arg z_1 \\ 0 & \text{if } \arg z_2 > \arg z_1 \end{cases}$$

Here we have observed wall-crossing phenomena of  $I_v(Z)$ , whose wall is given by

$$W = \{(z_1, z_2) \in \mathbb{H}^2 : \arg z_1 = \arg z_2\}.$$

**2.3. Weak stability conditions.** A slightly generalized notion of stability conditions is sometimes useful. For instance if we consider stability conditions in the sense of Definition 2.2, then there is no stability condition on  $\text{Coh}(X)$  if  $\dim X \geq 2$ . (cf. [36, Lemma 2.7].) On the other hand, there are classical notions of stability conditions on  $\text{Coh}(X)$ , such as slope stability. (cf. [16].) The slope stability can be formulated in the language of weak stability conditions introduced in [37].

Let  $\mathcal{A}$  be an abelian category. As in Subsection 2.1, we fix a finitely generated free abelian group  $\Gamma$  together with a group homomorphism  $\text{cl}: K(\mathcal{A}) \rightarrow \Gamma$ . We also fix a filtration of  $\Gamma$ ,

$$0 = \Gamma_{-1} \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_N = \Gamma,$$

such that each subquotient  $\Gamma_i/\Gamma_{i-1}$  is a free abelian group.

**Definition 2.4.** *A weak stability condition on  $\mathcal{A}$  is*

$$Z = \{Z_i\}_{i=0}^N \in \prod_{i=0}^N \text{Hom}_{\mathbb{Z}}(\Gamma_i/\Gamma_{i-1}, \mathbb{C}),$$

*such that the following conditions are satisfied:*

(i) *For non-zero  $E \in \mathcal{A}$ , take  $-1 \leq i \leq N$  such that  $\text{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$ . (We regard  $\Gamma_{-2} = \emptyset$ .) Then we have*

$$Z(E) := Z_i([\text{cl}(E)]) \in \mathbb{H}.$$

*Here  $[\text{cl}(E)]$  is the class of  $\text{cl}(E)$  in  $\Gamma_i/\Gamma_{i-1}$ . We say  $E \in \mathcal{A}$  is  $Z$ -(semi)stable if for any exact sequence  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  in  $\mathcal{A}$ , we have the inequality,*

$$(3) \quad \arg Z(F) < (\leq) \arg Z(G).$$

(ii) *There is a Harder-Narasimhan filtration for any  $E \in \mathcal{A}$ .*

When  $N = 0$ , a weak stability conditions is a stability condition in the sense of Definition 2.2.

**Remark 2.5.** *If the inequality (3) is strict, we have the following three possibilities:*

$$(4) \quad \arg Z(F) < \arg Z(E) < \arg Z(G),$$

$$(5) \quad \arg Z(F) < \arg Z(E) = \arg Z(G),$$

$$(6) \quad \arg Z(F) = \arg Z(E) < \arg Z(G).$$

*When  $N = 0$ , i.e.  $Z$  is a stability condition, then only the inequality (4) is possible. On the other hand when  $N > 0$ , the inequalities (5), (6) are also possible.*

Here we give some examples.



**Example 2.6.** (i) Let  $X$  be a  $d$ -dimensional smooth projective variety and  $\mathcal{A} = \text{Coh}(X)$ . Take  $\Gamma = \text{Im ch}$ ,  $\text{cl} = \text{ch}$  as in (1) and take a filtration

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_d,$$

given by

$$\Gamma_i = \Gamma \cap H^{\geq 2d-2i}(X, \mathbb{Q}).$$

Choose

$$0 < \phi_d < \phi_{d-1} < \cdots < \phi_0 < 1$$

and an ample divisor  $\omega$  on  $X$ . Set  $Z_i: \Gamma_i/\Gamma_{i-1} \rightarrow \mathbb{C}$  to be

$$Z_i(v) = \exp(\sqrt{-1}\pi\phi_i) \int_X v \cdot \omega^i.$$

Then  $Z = \{Z_i\}_{i=0}^d$  is a weak stability condition on  $\text{Coh}(X)$ . In this case,  $E \in \text{Coh}(X)$  is  $Z$ -semistable if and only if it is pure sheaf, i.e. there is no  $0 \neq F \subset E$  with  $\dim \text{Supp}(F) < \dim \text{Supp}(E)$ .

(ii) Let  $X$  be a smooth projective surface and take  $\Gamma$  and  $\text{cl}$  as above. We set  $\Gamma_0 \subset \Gamma_1 = \Gamma$  to be

$$\Gamma_0 = \Gamma \cap H^4(X, \mathbb{Q}),$$

hence

$$\Gamma_1/\Gamma_0 = \Gamma \cap (H^0 \oplus H^2).$$

We set  $Z_i: \Gamma_i/\Gamma_{i-1} \rightarrow \mathbb{C}$  to be

$$\begin{aligned} Z_0(n) &= -n, \\ Z_1(r, D) &= -D \cdot \omega + \sqrt{-1}r. \end{aligned}$$

Then  $Z = \{Z_i\}_{i=0}^1$  is a weak stability condition on  $\text{Coh}(X)$ . An object  $E \in \text{Coh}(X)$  is  $Z$ -semistable if and only if  $E$  is a torsion sheaf or an  $\omega$ -slope semistable sheaf. (cf. [16].)

In [37], the space of weak stability conditions on triangulated categories is introduced. Namely a weak stability condition on a triangulated category  $\mathcal{D}$  is a pair of  $(Z, \mathcal{A})$ , where  $\mathcal{A}$  is the heart of a bounded t-structure on  $\mathcal{D}$  and  $Z$  is a weak stability condition on  $\mathcal{A}$ . We denote by

$$(7) \quad \text{Stab}_{\Gamma_\bullet}(\mathcal{D}),$$

the set of weak stability conditions on  $\mathcal{D}$ , satisfying some good properties, i.e. local finiteness, support property. See [37, Section 2] for the detail on these properties. Using the same argument by Bridgeland [10, Theorem 7.1], it is proved in [37, Theorem 2.15] that the set (7) has a natural topology and each connected component is a complex manifold.

### 3. CURVE COUNTING INVARIANTS ON CALABI-YAU 3-FOLDS

In this section, we recall several curve counting theories on Calabi-Yau 3-folds, conjectures and the results. In what follows, we call a smooth projective complex 3-fold *Calab-Yau* if it satisfies the following condition,

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

For instance, the quintic 3-fold,

$$X = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4,$$

is a famous example of a Calabi-Yau 3-fold.

**3.1. Gromov-Witten theory.** Let  $X$  be a smooth projective Calabi-Yau 3-fold and  $C$  a connected 1-dimensional reduced  $\mathbb{C}$ -scheme with at worst nodal singularities. A morphism of schemes

$$f: C \rightarrow X,$$

is a *stable map* if the set of isomorphisms  $\phi: C \xrightarrow{\sim} C$  satisfying  $f \circ \phi = f$  is a finite set. This condition is equivalent to one of the following conditions.

- For any ample line bundle  $\mathcal{L}$  on  $X$ , the line bundle  $\omega_C \otimes f^* \mathcal{L}^{\otimes 3}$  is an ample line bundle on  $C$ . Here  $\omega_C$  is the dualizing sheaf of  $C$ .
- If  $C' \subset C$  is an irreducible component such that  $f(C')$  is a point, then

$$2g(C') + \sharp \left( C' \cap (\overline{C \setminus C'}) \right) \geq 3.$$

Here  $g(*)$  is the arithmetic genus. The moduli space of such maps is constructed after we fix the following numerical data,

$$g \in \mathbb{Z}_{\geq 0}, \quad \beta \in H_2(X, \mathbb{Z}).$$

We call a stable map  $(C, f)$  as *type*  $(g, \beta)$  if  $g(C) = g$  and the map  $f$  satisfies  $f_*[C] = \beta$ . The moduli space of stable maps  $(C, f)$  of type  $(g, \beta)$  is denoted by,

$$(8) \quad \overline{M}_g(X, \beta).$$

The moduli space (8) is a Deligne Mumford stack of finite type over  $\mathbb{C}$  [22]. However the space (8) may be singular and its dimension may be different from its expected dimension. In fact the tangent space and the obstruction space of the space of maps  $f: C \rightarrow X$  for a fixed  $C$  are given by

$$H^0(C, f^* T_X), \quad H^1(C, f^* T_X),$$

respectively. Hence the expected dimension of the space (8) is

$$\begin{aligned} & \chi(C, f^*T_X) + \dim \overline{M}_g \\ &= \frac{3}{2} \deg T_C + 3g - 3 \\ &= 0. \end{aligned}$$

Here  $\overline{M}_g$  is the moduli space of genus  $g$  stable curves. Here we have used the Riemann-Roch theorem on  $C$  and the Calabi-Yau assumption of  $X$ .

Now there is a way to construct the 0-dimensional virtual fundamental cycle on (8) via perfect obstruction theory [4], [28]. By definition, a *perfect obstruction theory* on a scheme (or Deligne-Mumford stack)  $M$  is a morphism in the derived category of coherent sheaves  $D^b \text{Coh}(M)$ ,

$$(9) \quad h: E^\bullet \rightarrow L_M,$$

where  $E^\bullet$  is a complex of vector bundles on  $M$  concentrated on  $[-1, 0]$  and  $L_M$  is the cotangent complex of  $M$ . The morphism  $h$  should satisfy that  $h^0$  is an isomorphism and  $h^{-1}$  is surjective. Given such a morphism (9), we are able to construct the virtual fundamental cycle,

$$[M]^{\text{vir}} \in A_{\text{rank } E^0 - \text{rank } E^{-1}}(M).$$

Here  $A_*(M)$  is the Chow group of  $M$ . Roughly speaking, the cycle  $[M]^{\text{vir}}$  is constructed by taking the intersection of the intrinsic normal cone and the 0-section in the vector bundle stack  $[(E^{-1})^\vee / (E^0)^\vee]$ . (See [4], [28] for the detail.)

By [4], [28], there is a perfect obstruction theory on the moduli space (8). The resulting virtual fundamental cycle is denote by

$$[\overline{M}_g(X, \beta)]^{\text{vir}} \in A_0(\overline{M}_g(X, \beta), \mathbb{Q}).$$

Integrating the virtual cycle, we obtain the GW invariant.

**Definition 3.1.** The *Gromov-Witten (GW) invariant* is defined by

$$N_{g, \beta}^{\text{GW}} = \int_{[\overline{M}_g(X, \beta)]^{\text{vir}}} 1 \in \mathbb{Q}.$$

**Remark 3.2.** Since  $\overline{M}_g(X, \beta)$  is not a scheme but a Deligne-Mumford stack, the resulting invariant  $N_{g, \beta}^{\text{GW}}$  is not an integer in general.

One of the important examples is a contribution of multiple covers to a fixed super rigid rational curve.

**Example 3.3.** Let

$$f: X \rightarrow Y,$$

be a birational contraction which contracts a smooth super rigid rational curve  $C \subset X$ , i.e.

$$N_{C/X} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

In this case, the computation of  $N_{g,d[C]}^{\text{GW}}$  can be reduced to a certain integration over the space  $\overline{M}_g(\mathbb{P}^1, d)$ . We have the following diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & \mathbb{P}^1, \\ \pi \downarrow & & \\ \overline{M}_g(\mathbb{P}^1, d) & & \end{array}$$

where  $\pi$  is the universal curve and  $\phi$  is the universal morphism. Then we have

$$(10) \quad N_{g,d[C]}^{\text{GW}} = \int_{[\overline{M}_g(\mathbb{P}^1, d)]^{\text{vir}}} c_{\text{top}}(R^1\pi_*\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}).$$

The invariants (10) are computed in [12],

$$\begin{aligned} N_{0,d[C]}^{\text{GW}} &= \frac{1}{d^3}, \quad N_{1,d[C]}^{\text{GW}} = \frac{1}{12d}, \\ N_{g,d[C]}^{\text{GW}} &= \frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!}, \quad g \geq 2. \end{aligned}$$

Here  $B_{2g}$  is the  $2g$ -th Bernoulli number.

**3.2. Donaldson-Thomas theory.** Another curve counting invariant on a Calabi-Yau 3-fold  $X$  is defined by the integration of the virtual fundamental cycle on the moduli space of subschemes,

$$(11) \quad Z \subset X,$$

satisfying  $\dim Z \leq 1$ . Given a numerical data,

$$n \in \mathbb{Z}, \quad \beta \in H_2(X, \mathbb{Z}),$$

the relevant moduli space is the classical Hilbert scheme,

$$(12) \quad \text{Hilb}_n(X, \beta),$$

which parameterizes subschemes (11) satisfying

$$(13) \quad \chi(\mathcal{O}_Z) = n, \quad [Z] = \beta.$$

Recall that the moduli space (12) is a projective scheme.

The moduli space (12) is also interpreted as a moduli space of rank one torsion free sheaves on  $X$  with a trivial first Chern class. Namely if  $I$  is a torsion free sheaf of rank one, then  $I$  fits into the exact sequence,

$$0 \rightarrow I \rightarrow I^{\vee\vee} \rightarrow F \rightarrow 0,$$

such that  $F$  is one or zero dimensional sheaf. It can be shown that  $I^{\vee\vee}$  is a line bundle on  $X$ , hence isomorphic to  $\mathcal{O}_X$  if its first Chern class is zero. Hence  $I$  is isomorphic to  $I_Z$ , the ideal sheaf of a subscheme  $Z \subset X$

with  $\dim Z \leq 1$ . The condition (13) is equivalent to the condition on the Chern character,

$$(14) \quad \text{ch}(I_Z) = (1, 0, -\beta, -n)$$

$$(15) \quad \in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \oplus H^6(X, \mathbb{Z}).$$

Here we have regarded  $\beta$  and  $n$  as elements of  $H^4(X, \mathbb{Z})$  and  $H^6(X, \mathbb{Z})$  by the Poincaré duality. As a summary, there is a one to one correspondence between subschemes (11) satisfying (13) and torsion free sheaves  $I$  on  $X$  satisfying (14), via  $Z \mapsto I_Z$ .

If we regard the space (12) as a moduli space of rank one torsion free sheaves, the deformation theory of coherent sheaves implies that the spaces

$$\text{Ext}_X^1(I_Z, I_Z), \quad \text{Ext}_X^2(I_Z, I_Z),$$

are tangent space and the obstruction space at the point  $[Z] \in \text{Hilb}_n(X, \beta)$  respectively. Since  $X$  is a Calabi-Yau 3-fold, the Serre duality implies that

$$\text{Ext}_X^2(I_Z, I_Z) \cong \text{Ext}_X^1(I_Z, I_Z)^\vee.$$

In particular the expected dimension of the space (12) is

$$\dim \text{Ext}_X^1(I_Z, I_Z) - \dim \text{Ext}_X^2(I_Z, I_Z) = 0.$$

In fact there is a perfect obstruction theory on  $\text{Hilb}_n(X, \beta)$ , (cf. [34],)

$$E^\bullet \rightarrow L_{\text{Hilb}_n(X, \beta)},$$

satisfying that

$$(16) \quad E^\bullet \cong E^{\bullet\vee}[1].$$

A perfect obstruction theory satisfying the symmetry (16) is called a *perfect symmetric obstruction theory*. We have the associated virtual fundamental cycle,

$$[\text{Hilb}_n(X, \beta)]^{\text{vir}} \in A_0(\text{Hilb}_n(X, \beta), \mathbb{Z}).$$

The DT invariant is defined by the integration over the virtual fundamental cycle.

**Definition 3.4.** The *Donaldson-Thomas (DT) invariant* is defined by

$$(17) \quad I_{n, \beta} = \int_{[\text{Hilb}_n(X, \beta)]^{\text{vir}}} 1 \in \mathbb{Z}.$$

So far,  $I_{n, \beta}$  are computed in several examples in terms of generating functions.

**Example 3.5.** (i) In the case of  $\beta = 0$ , the generating series of  $I_{n,0}$  is computed by Li [27], Behrend-Fantechi [5] and Levine-Pandharipande [26],

$$\sum_{n \in \mathbb{Z}} I_{n,0} q^n = M(-q)^{\chi(X)}.$$

Here  $M(q)$  is the MacMahon function,

$$\begin{aligned} M(q) &= \prod_{k \geq 1} \frac{1}{(1 - q^k)^k} \\ &= 1 + q + 3q^2 + 6q^3 + \cdots. \end{aligned}$$

(ii) Let  $C \subset X$  is a super rigid rational curve as in Example 3.3. Then the invariant  $I_{n,d[C]}$  is computed by Behrend-Bryan [3],

$$\sum_{n,d} I_{n,d[C]} q^n t^d = M(-q)^{\chi(X)} \prod_{k \geq 1} (1 - (-q)^k t)^k.$$

**3.3. DT theory via Behrend function.** The integration (17) is usually difficult to compute. On the other hand, Behrend [2] shows that the invariant (17) is also obtained as a certain weighted Euler characteristic of a certain constructible function on  $\text{Hilb}_n(X, \beta)$ . In many situations, computations of weighted Euler characteristic are easier than computations of virtual fundamental cycles.

In fact for any  $\mathbb{C}$ -scheme  $M$ , Behrend [2] constructs a canonical constructible function,

$$\nu_M: M \rightarrow \mathbb{Z},$$

satisfying the following properties.

- If  $\pi: M_1 \rightarrow M_2$  is a smooth morphism with relative dimension  $d$ , we have

$$\nu_{M_1} = (-1)^d \pi^* \nu_{M_2}.$$

- For  $p \in M$ , suppose that there is an analytic open neighborhood  $p \in U \subset M$ , a complex manifold  $V$  and a holomorphic function  $f: V \rightarrow \mathbb{C}$  such that  $U \cong \{df = 0\}$ . Then we have

$$(18) \quad \nu(p) = (-1)^{\dim V} (1 - \chi(M_p(f))).$$

Here  $M_p(f)$  is the Milnor fiber of  $f$  at  $p \in V$ .

- If  $M$  has a symmetric perfect obstruction theory, we have

$$\begin{aligned} (19) \quad \int_{[M]^{\text{vir}}} 1 &= \int_M \nu_M d\chi, \\ &:= \sum_{k \in \mathbb{Z}} k \chi(\nu^{-1}(k)). \end{aligned}$$

Here the Milnor fiber  $M_p(f)$  is defined as follows. Let  $p \in V' \subset V$  be an analytic small neighborhood and fix a norm  $\|*\|$  on  $V'$ . Then for  $0 < \varepsilon \ll \delta \ll 1$ , the topological type of the space

$$(20) \quad \{z \in V' : \|z - p\| \leq \delta, f(z) = f(p) + \varepsilon\},$$

does not depend on  $\varepsilon, \delta$ . The Milnor fiber  $M_p(f)$  is defined to be the topological space (20).

By the property (19), the invariant  $I_{n,\beta}$  is also obtained by

$$I_{n,\beta} = \int_{\text{Hilb}_n(X,\beta)} \nu d\chi.$$

Here we have written  $\nu_{\text{Hilb}_n(X,\beta)}$  as  $\nu$  for simplicity. An important fact is that the local moduli space of objects in  $\text{Coh}(X)$  is analytically locally written as a critical locus of some holomorphic function on a complex manifold up to gauge equivalence. This fact is proved in [19, Theorem 5.2] in a more general setting. In particular the function  $\nu$  on  $\text{Hilb}_n(X, \beta)$  can be computed using the expression (18).

A rough idea of the proof of the critical locus condition in [19, Theorem 5.2] is as follows: for  $E \in \text{Coh}(X)$ , we are interested in the deformations of  $E$ . By applying spherical twists associated to line bundles, we may assume that  $E$  is a locally free sheaf, or equivalently a holomorphic vector bundle. (cf. [19, Corollary 8.5].) Let

$$\bar{\partial}: E \rightarrow E \otimes \Omega^{0,1},$$

be the  $\bar{\partial}$ -connection which determines a holomorphic structure of  $E$ , where  $\Omega^{0,1}$  is the sheaf of  $(0,1)$ -forms of  $X$ . Then giving a deformation of  $E$  is equivalent to giving a deformation of  $\bar{\partial}$  up to gauge equivalence. This is equivalent to giving

$$A \in A^{0,1}(X, \mathcal{E}nd(E)),$$

where  $A^{0,1}(X, \mathcal{E}nd(E))$  is the space of  $\mathcal{E}nd(E)$ -valued  $(0,1)$  forms, satisfying

$$(21) \quad (\bar{\partial} + A)^2 = 0,$$

up to gauge equivalence. The equation (21) is equivalent to

$$\bar{\partial}A + A \wedge A = 0.$$

Let CS be the *holomorphic Chern Simons function*,

$$\text{CS}: A^{0,1}(X, \mathcal{E}nd(E)) \rightarrow \mathbb{C},$$

defined by

$$\text{CS}(A) = \int_X \left( \frac{1}{2} \bar{\partial}A \wedge A + \frac{1}{3} A \wedge A \wedge A \right) \wedge \sigma_X,$$

where  $\sigma_X$  is a no-where vanishing holomorphic 3-form on  $X$ . (cf. [34].) Then  $A \in A^{0,1}(X, \mathcal{E}nd(E))$  satisfies the equation (21) if and only if  $A$

is a critical locus of the function CS. Therefore the local moduli space of  $E$  is written as

$$\{d\text{CS} = 0\}/G,$$

where  $G$  is the group of isomorphisms of  $E$  as a  $C^\infty$ -vector bundle, i.e. the local moduli space of objects in  $\text{Coh}(X)$  is written as a critical locus up to gauge equivalence.

However  $A^{0,1}(X, \mathcal{E}nd(E))$  is an infinite dimensional vector space, and we need to find suitable finite dimensional vector subspace of  $A^{0,1}(X, \mathcal{E}nd(E))$ . This is worked out in [19, Theorem 5.2] by using the Hodge theory. Namely the space of harmonic forms  $U$  on  $A^{0,1}(X, \mathcal{E}nd(E))$  is finite dimensional, satisfying  $U \cong \text{Ext}^1(E, E)$ , and we restrict CS to  $U$ . For the detail, see [19, Theorem 5.2].

**Example 3.6.** (i) Suppose that  $\text{Hilb}_n(X, \beta)$  is non-singular of dimension  $d$ . By the property (18), the Behrend function on  $\text{Hilb}_n(X, \beta)$  coincides with  $(-1)^d$ . Therefore we have

$$I_{n,\beta} = (-1)^d \chi(\text{Hilb}_n(X, \beta)).$$

(ii) Suppose that  $\text{Hilb}_n(X, \beta)$  is isomorphic to the spectrum of  $\mathbb{C}[z]/z^k$  for some  $k \geq 1$ . (For instance, the local moduli space of a rigid rational curve  $C \subset X$  with  $N_{C/X} = \mathcal{O}_C \oplus \mathcal{O}_C(-2)$  is written as the spectrum of  $\mathbb{C}[z]/z^k$  for some  $k \geq 1$ .) Then  $\text{Hilb}_n(X, \beta)$  is written as  $\{df = 0\}$ , where  $f$  is

$$f: \mathbb{C} \ni z \mapsto z^{k+1} \in \mathbb{C}.$$

The Milnor fiber of  $f$  at  $0 \in \mathbb{C}$  is  $(k+1)$ -points, hence we have

$$I_{n,\beta} = \nu(0) = k.$$

**3.4. GW/DT correspondence.** As we mentioned before, GW invariant is not necessary an integer while DT invariant is always an integer. Although both theories seem different, Maulik-Nekrasov-Okounkov-Pandharipande [30] propose a conjecture on a certain relationship between GW and DT theories. The conjecture is formulated in terms of generating functions, and it also implies a hidden integrality of GW invariants.

Let us introduce the generating functions. The generating function of GW side is

$$\text{GW}(X) = \sum_{g \geq 0, \beta > 0} N_{g,\beta}^{\text{GW}} \lambda^{2g-2} t^\beta.$$

Here  $\beta > 0$  means  $\beta$  is a homology class of a non-zero effective one cycle on  $X$ . Similarly the generating function of DT side is

$$\text{DT}(X) = \sum_{n \in \mathbb{Z}, \beta \geq 0} I_{n,\beta} q^n t^\beta.$$



The series  $\mathrm{DT}(X)$  can be written as

$$\mathrm{DT}(X) = \sum_{\beta \geq 0} \mathrm{DT}_{\beta}(X) t^{\beta},$$

where  $\mathrm{DT}_{\beta}(X)$  is a Laurent series of  $q$ . (It is easy to check that  $\mathrm{Hilb}_n(X, \beta) = \emptyset$ , hence  $I_{n, \beta} = 0$ , for  $n \ll 0$ .) The term  $\mathrm{DT}_0(X)$  is a contribution of zero dimensional subschemes, and does not contribute to curve counting on  $X$ . The *reduced DT series* are defined by

$$(22) \quad \mathrm{DT}'(X) = \frac{\mathrm{DT}(X)}{\mathrm{DT}_0(X)}, \quad \mathrm{DT}'_{\beta}(X) = \frac{\mathrm{DT}_{\beta}(X)}{\mathrm{DT}_0(X)}.$$

Note that  $\mathrm{DT}_0(X)$  is given by the power of the MacMahon function by Example 3.5 (i).

**Conjecture 3.7.** [30]

(i) (**Rationality conjecture**): *The Laurent series  $\mathrm{DT}'_{\beta}(X)$  is the Laurent expansion of a rational function of  $q$ , invariant under  $q \leftrightarrow 1/q$ .*

(ii) (**GW/DT correspondence**): *By the variable change  $q = -e^{i\lambda}$ , we have the equality of the generating series,*

$$\exp \mathrm{GW}(X) = \mathrm{DT}'(X).$$

Here we need some explanation on the above conjecture. The series  $\mathrm{DT}'_{\beta}(X)$  is a priori a Laurent series of  $q$  and it is not obvious whether it converges or not near  $q = 0$ . The rationality conjecture asserts that  $\mathrm{DT}'_{\beta}(X)$  actually converges near  $q = 0$ , and moreover it can be analytically continued to give a meromorphic function (in fact rational function) on the  $q$ -plane. The invariance under  $q \leftrightarrow 1/q$  implies that the above analytic continuation satisfies the automorphic property with respect the transformation  $q \leftrightarrow 1/q$ . For instance in the situation of Example 3.3, the series  $\mathrm{DT}'_{[C]}(X)$  is

$$(23) \quad \begin{aligned} \mathrm{DT}'_{[C]}(X) &= q - 2q^2 + 3q^3 - \cdots, \\ &= \frac{q}{(1+q)^2}. \end{aligned}$$

The rational function (23) is invariant under  $q \leftrightarrow 1/q$ .

If we assume the rationality conjecture, we can expand  $\mathrm{DT}'(X)$  near  $q = -1$ , and write it by the  $\lambda$ -variable via  $q = -e^{i\lambda}$ . The invariance of  $\mathrm{DT}'_{\beta}(X)$  under  $q \leftrightarrow 1/q$  implies that  $i$  is not involved in the  $\lambda$ -expansion. The GW/DT correspondence asserts that the coefficients of the above expansion are described in terms of GW invariants.

So far the above conjecture has been checked in several situations. For instance the GW/DT correspondence for a local  $(-1, -1)$ -curve can be checked from Example 3.3 and Example 3.5, as discussed in [3]. Also GW/DT correspondence for toric Calabi-Yau 3-folds and local curves are proved in [30] and [31] respectively, by using torus localization and degeneration formula. On the other hand, at this moment,

these arguments are applied to the above specific examples, and not to arbitrary Calabi-Yau 3-folds. We have few tools in approaching Conjecture 3.7 in a general setting, except the recent progress of *wall-crossing formula* of DT type invariants. This is established by Joyce-Song [19] and Kontsevich-Soibelman [23], and is an effective tool in studying DT type curve counting invariants for arbitrary Calabi-Yau 3-folds. So far, several applications have been given, including Conjecture 3.7 (i).

A rough idea of the application of the wall-crossing formula is as follows. Recall that the moduli space  $\text{Hilb}_n(X, \beta)$  is interpreted as a moduli space of torsion free rank one sheaves on  $X$ . This is nothing but the moduli space of stable objects on  $\text{Coh}(X)$  w.r.t. weak stability conditions in Example 2.6 (i). One may try to change weak stability conditions on  $\text{Coh}(X)$ , construct other DT type invariants counting stable objects, and see wall-crossing phenomena as we discussed in Subsection 2.2. However we can easily see that there is no interesting wall-crossing phenomena w.r.t weak stability conditions constructed in Example 2.6 (i). Instead we can also study (weak) stability conditions on other abelian subcategory in the derived category of coherent sheaves  $D^b \text{Coh}(X)$ , e.g. the heart of a bounded t-structure on  $D^b \text{Coh}(X)$ . Then we can construct DT type invariants counting stable objects in the derived category, and the wall-crossing formula describes how these invariants vary under change of (weak) stability conditions. If we choose some specific (weak) stability condition, then the generating series sometimes becomes simpler than the original DT series, thus giving some non-trivial result to the DT series.

As we mentioned, an important point is that the wall-crossing formula is applied for any Calabi-Yau 3-fold, and not restricted to specific examples, e.g. toric Calabi-Yau 3-folds. Using this new kind of technology, Conjecture 3.7 (i) is now solved.<sup>1</sup> We will discuss this more in Subsection 3.6 below.

**3.5. Pandharipande-Thomas theory.** Another application of the wall-crossing formula is the so called DT/PT correspondence, that is the correspondence between DT invariants and invariants counting stable pairs [32]. The notion of stable pairs is introduced by Pandharipande-Thomas [32] in order to give a geometric understanding of the reduced DT theory (22). By definition, a *stable pair* on a Calabi-Yau 3-fold  $X$  is a pair

$$(F, s),$$

where  $F$  is a coherent sheaf on  $X$  and  $s: \mathcal{O}_X \rightarrow F$  is a morphism satisfying the following.

- $F$  is a pure one dimensional sheaf, i.e. there is no zero dimensional subsheaf in  $F$ .

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<sup>1</sup>We need the result of [6] which is not yet written at this moment.

- The cokernel of  $s$  is a zero dimensional sheaf.

For instance let  $C \subset X$  be a smooth curve and  $D \subset C$  a divisor on  $C$ . We set  $F = \mathcal{O}_C(D)$  and define the morphism  $s$  to be the composition,

$$s: \mathcal{O}_X \rightarrow \mathcal{O}_C \hookrightarrow \mathcal{O}_C(D).$$

Then the pair  $(F, s)$  is a stable pair. As the above example indicates, roughly speaking, a stable pair is a pair of a curve on  $X$  and an effective divisor on it.

Note that if  $Z \subset X$  is a subscheme giving a point in  $\text{Hilb}_n(X, \beta)$ , we have a pair

$$(\mathcal{O}_Z, s), \quad s: \mathcal{O}_X \rightarrow \mathcal{O}_Z,$$

where  $s$  is a natural surjection. The pair  $(\mathcal{O}_Z, s)$  fails to be a stable pair if and only if  $\mathcal{O}_Z$  contains a zero dimensional subsheaf. On the other hand, a stable pair  $(F, s)$  determines a point in  $\text{Hilb}_n(X, \beta)$  if and only if  $s$  is surjective.

Similarly to the DT theory, we consider the moduli space of stable pairs  $(F, s)$  satisfying

$$[F] = \beta, \quad \chi(F) = n.$$

Here  $[F]$  is the fundamental homology class determined by the one dimensional sheaf  $F$ . The resulting moduli space is denoted by

$$(24) \quad P_n(X, \beta).$$

The moduli space (24) is proved to be a projective scheme in [32]. Moreover the space (24) is interpreted as a moduli space of two term complexes,

$$(25) \quad I^\bullet = \cdots \rightarrow 0 \rightarrow \mathcal{O}_X \xrightarrow{s} F \rightarrow 0 \rightarrow \cdots,$$

in the derived category of coherent sheaves, i.e.

$$I^\bullet \in D^b \text{Coh}(X).$$

The deformation theory of objects in the derived category yields that the spaces

$$\text{Ext}_X^1(I^\bullet, I^\bullet), \quad \text{Ext}_X^2(I^\bullet, I^\bullet),$$

are tangent space and the obstruction space respectively, which are dual by the Serre duality. Similarly to the DT theory, the above deformation theory provides a perfect symmetric obstruction theory on the space (24), hence the 0-dimensional virtual cycle.

**Definition 3.8.** The *Pandharipande-Thomas (PT) invariant* is defined by

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Z}.$$

As in the DT case, the invariant  $P_{n,\beta}$  is also defined by

$$P_{n,\beta} = \int_{P_n(X,\beta)} \nu d\chi,$$

for the Behrend function,

$$\nu: P_n(X, \beta) \rightarrow \mathbb{Z}.$$

**Example 3.9.** Let  $C \cong \mathbb{P}^1 \subset X$  be a super rigid rational curve as in Example 3.3. Then  $(F, s)$  is a stable pair with  $[F] = [C]$  and  $\chi(F) = n$  if and only if

$$F = \mathcal{O}_C(n-1), \quad s \in H^0(C, \mathcal{O}_C(n-1)) \setminus \{0\}.$$

Hence we have,

$$\begin{aligned} P_n(X, [C]) &\cong \mathbb{P}(H^0(C, \mathcal{O}_C(n-1))), \\ &\cong \mathbb{P}^{n-1}. \end{aligned}$$

Therefore we have

$$\begin{aligned} P_{n,[C]} &= (-1)^{\dim P_n(X,[C])} \chi(P_n(X, [C])), \\ &= (-1)^{n-1} n. \end{aligned}$$

The generating series is

$$\sum_{n \in \mathbb{Z}} P_{n,[C]} q^n = q - 2q^2 + 3q^3 - \cdots.$$

Note that the above series coincides with  $\text{DT}'_{[C]}(X)$  by (23).

Similarly to the DT theory, we consider the generating series,

$$\begin{aligned} \text{PT}(X) &= \sum_{n \in \mathbb{Z}, \beta \geq 0} P_{n,\beta} q^n t^\beta \\ &= 1 + \sum_{\beta > 0} \text{PT}_\beta(X), \end{aligned}$$

where  $\text{PT}_\beta(X)$  is a Laurent series of  $q$ . In [32], Pandharipande-Thomas propose the following conjecture.

**Conjecture 3.10.** We have the equality of the generating series,

$$(26) \quad \text{DT}'_\beta(X) = \text{PT}_\beta(X).$$

Note that we have already observed the formula (26) in Example 3.9 when the curve class is a class of a super rigid rational curve.

Similarly to Conjecture 3.7 (i), the formula (26) is also a consequence of the wall-crossing formula. A rough idea is as follows. Suppose that there is an abelian subcategory  $\mathcal{A}$  in  $D^b \text{Coh}(X)$  and a stability condition  $\sigma$  on it, such that the ideal sheaf  $I_Z$  for a 1-dimensional subscheme  $Z \subset X$  is a  $\sigma$ -stable object in  $\mathcal{A}$ . If there is a 0-dimensional

subsheaf  $Q \subset \mathcal{O}_Z$ , i.e.  $\mathcal{O}_X \rightarrow \mathcal{O}_Z$  is not a stable pair, then there is a sequence,

$$(27) \quad Q[-1] \rightarrow I_Z \rightarrow I_{Z'},$$

where  $Z'$  is a 1-dimensional subscheme in  $Z$  defined by  $\mathcal{O}_{Z'} = \mathcal{O}_Z/Q$ . Suppose that the sequence (27) is an exact sequence in  $\mathcal{A}$ . Then we expect that we can deform a stability condition  $\sigma$  to another stability condition  $\tau$  such that the sequence (27) destabilizes  $I_Z$  w.r.t.  $\tau$ . Instead if we take an exact sequence in  $\mathcal{A}$ ,

$$I_{Z'} \rightarrow E \rightarrow Q[-1],$$

then the object  $E$  may be  $\tau$ -stable. Such an object  $E$  is isomorphic to a two term complex,

$$E \cong (\mathcal{O}_X \xrightarrow{s} F),$$

for a one dimensional sheaf  $F$ , and one may expect that  $(F, s)$  is a stable pair. If the above story is correct, then  $\sigma$  corresponds to the DT theory,  $\tau$  corresponds to the PT theory, and the relationship between these theories should be described by the wall-crossing formula.

**3.6. Product formula of the generating series.** In this subsection, we discuss the result obtained by applying the wall-crossing formula.

**Theorem 3.11.** [38], [37], [9] *For each  $n \in \mathbb{Z}$  and  $\beta \in H_2(X, \mathbb{Z})$ , there are invariants,*

$$N_{n,\beta} \in \mathbb{Q}, \quad L_{n,\beta} \in \mathbb{Q},$$

satisfying that

- there is  $d \in \mathbb{Z}_{>0}$  such that  $N_{n,\beta} = N_{n',\beta}$  if  $n \pm n' \in d\mathbb{Z}$  and  $\beta \neq 0$ .
- $L_{n,\beta} = L_{-n,\beta}$ , and  $L_{n,\beta} = 0$  for  $|n| \gg 0$ ,

such that we have the following infinite product expansion formula,

$$(28) \quad \text{PT}(X) = \prod_{n>0, \beta>0} \exp((-1)^{n-1} n N_{n,\beta} q^n t^\beta) \left( \sum_{n,\beta} L_{n,\beta} q^n t^\beta \right),$$

$$(29) \quad \text{DT}(X) = \prod_{n>0} \exp((-1)^{n-1} n N_{n,0} q^n) \text{PT}(X).$$

We will explain how to deduce the formula (28) via wall-crossing in Section 5.

**Remark 3.12.** *More precisely, the results in [38], [37] are Euler characteristic versions of the corresponding results, i.e. take the (non-weighted) Euler characteristic in defining the invariants  $I_{n,\beta}$ ,  $P_{n,\beta}$ . As discussed in the arXiv version of [37, Theorem 8.11], the formulas (28), (29) can be proved by combining the work of Joyce-Song [19] and Behrend-Getzler's announced result [6]. The latter result is the*

derived category version of [19, Theorem 5.3], that is the moduli stack of certain objects in the derived category is locally written as a critical locus of some holomorphic function up to gauge action. The precise statement is formulated in [35, Conjecture 4.3]. On the other hand, in [9], Bridgeland proves Theorem 3.11 without relying [6], using the arguments different to ours.

The invariants  $N_{n,\beta}$  and  $L_{n,\beta}$  are also interpreted as counting invariants of certain objects in the derived category. Roughly speaking:

- Let  $\omega$  be an  $\mathbb{R}$ -ample divisor and  $Z_\omega$  the stability condition on  $\text{Coh}_{\leq 1}(X)$  constructed in Example 2.3 (iii). In the notation of Example 2.3 (iii), the invariant  $N_{n,\beta}$  counts  $Z_\omega$ -semistable objects  $E \in \text{Coh}_{\leq 1}(X)$ , satisfying

$$\text{cl}_0(E) = (n, \beta) \in \Gamma_0.$$

- The invariant  $L_{n,\beta}$  counts certain semistable objects in the derived category  $E \in D^b \text{Coh}(X)$ , satisfying

$$\begin{aligned} \text{ch}(E) &= (1, 0, -\beta, -n) \\ &\in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \oplus H^6(X, \mathbb{Z}). \end{aligned}$$

The relevant stability condition is self dual w.r.t. the derived dual.

In order to define  $N_{n,\beta}$ , we need to choose an  $\mathbb{R}$ -ample divisor  $\omega$ , but it can be shown that  $N_{n,\beta}$  does not depend on  $\omega$ . (cf. Lemma 4.8.) The self duality in defining  $L_{n,\beta}$  means that, if  $E$  is (semi)stable, then its derived dual

$$\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X) \in D^b \text{Coh}(X),$$

is also (semi)stable. The equality  $L_{n,\beta} = L_{-n,\beta}$  is a consequence of the self duality.

In some cases, the invariants  $N_{n,\beta}$  and  $L_{n,\beta}$  are defined in a similar way to DT or PT invariants. Let us take  $n \in \mathbb{Z}$ ,  $\beta \in H_2(X, \mathbb{Z})$  and an ample  $\mathbb{R}$ -divisor  $\omega$  on  $X$ . Let  $M_{n,\beta}(\omega)$  be the moduli space of  $Z_\omega$ -semistable objects  $E \in \text{Coh}_{\leq 1}(X)$  satisfying  $\text{cl}_0(E) = (n, \beta)$ , in the notation of Example 2.3 (iii). If  $n$  and  $\beta$  are coprime and  $\omega$  is in a general position of the ample cone, then any  $Z_\omega$ -semistable sheaf  $E \in \text{Coh}_{\leq 1}(X)$  is  $Z_\omega$ -stable, and the moduli space  $M_{n,\beta}(\omega)$  is a projective scheme with a symmetric perfect obstruction theory. The invariant  $N_{n,\beta}(\omega)$  is defined by

$$\begin{aligned} N_{n,\beta}(\omega) &:= \int_{[M_{n,\beta}(\omega)]^{\text{vir}}} 1 \\ &= \int_{M_{n,\beta}(\omega)} \nu d\chi. \end{aligned}$$

Here  $\nu$  is the Behrend function on  $M_{n,\beta}(\omega)$ . We will see in Lemma 4.8 that  $N_{n,\beta}(\omega)$  is independent of  $\omega$ , so we can write it as  $N_{n,\beta}$ .

On the other hand if  $n$  and  $\beta$  are not coprime, then  $Z_\omega$ -semistable sheaf may not be  $Z_\omega$ -stable, and there is no fine moduli space  $M_{n,\beta}(\omega)$  in this case. Instead we should work with the moduli stack of  $Z_\omega$ -semistable objects, denoted by  $\mathcal{M}_{n,\beta}(\omega)$ . The moduli stack  $\mathcal{M}_{n,\beta}(\omega)$  is known to be an Artin stack of finite type over  $\mathbb{C}$ . However it is not obvious how to define counting invariants via  $\mathcal{M}_{n,\beta}(\omega)$ , since at this moment there is no reasonable notion of perfect obstruction theories nor virtual fundamental cycles on Artin stacks. Also it is not obvious how to define the weighted Euler characteristic of  $\mathcal{M}_{n,\beta}(\omega)$ , weighted by the Behrend function. The only known way (at this moment) to do this is to introduce the ‘logarithm’ of the moduli stack  $\mathcal{M}_{n,\beta}(\omega)$  in the Hall algebra and integrate it. We will discuss this construction in Section 4.

As a corollary of Theorem 3.11, we have the following result.

**Corollary 3.13.** [38], [37], [9] *Conjecture 3.7 (i) and Conjecture 3.10 are true.*

*Proof.* The property of  $N_{n,\beta}$  easily implies that the series

$$(30) \quad \sum_{n>0} (-1)^{n-1} n N_{n,\beta} q^n,$$

is the Laurent expansion of a rational function of  $q$ , invariant under  $q \leftrightarrow 1/q$ . (cf. [38, Lemma 4.6].) Then Conjecture 3.7 (i) follows from the rationality of (30) and the property of  $L_{n,\beta}$ .

As for Conjecture 3.10, the formula (29) in particular implies that

$$\mathrm{DT}_0(X) = \prod_{n>0} \exp((-1)^{n-1} n N_{n,0} q^n).$$

Hence the formula (26) follows.  $\square$

#### 4. HALL ALGEBRAS AND GENERALIZED DONALDSON-THOMAS INVARIANTS

In Subsection 3.6, we have discussed the invariants  $N_{n,\beta}$  and  $L_{n,\beta}$ , which count certain objects the derived category  $D^b \mathrm{Coh}(X)$ . As we discussed there, the definition of these invariants is not obvious if there is a strictly semistable object. In this section, we introduce (stack theoretic) Hall algebra of coherent sheaves, and explain how to construct  $N_{n,\beta}$  via that algebra. The construction is due to Joyce-Song [19], which is called *generalized Donaldson-Thomas invariant*. (The invariant  $L_{n,\beta}$  can be similarly constructed, and we will discuss it later in Section 5.)

**4.1. Grothendieck groups of varieties.** We recall the notion of Grothendieck groups of varieties. Let  $S$  be a variety over  $\mathbb{C}$ . We define the group  $K(\text{Var}/S)$  to be the group generated by isomorphism classes of symbols

$$[\rho: Y \rightarrow S],$$

where  $\rho: Y \rightarrow S$  is an  $S$ -variety of finite type over  $\mathbb{C}$ , and two symbols  $[\rho_i: Y_i \rightarrow S]$  for  $i = 1, 2$  are isomorphic if there is an isomorphism  $Y_1 \xrightarrow{\sim} Y_2$  preserving the morphisms  $\rho_i$ . The relation is generated by

$$[\rho: Y \rightarrow S] \sim [\rho|_V: V \rightarrow S] + [\rho|_U: U \rightarrow S],$$

where  $V \subset Y$  is a closed subvariety and  $U := Y \setminus V$ . If  $S = \text{Spec } \mathbb{C}$ , we write  $K(\text{Var}/S)$  as  $K(\text{Var}/\mathbb{C})$  for simplicity.

The structure of the group  $K(\text{Var}/\mathbb{C})$  is studied in [7]. This is generated by smooth projective varieties  $[Y]$  with relation given by

$$(31) \quad [\widehat{Y}] - [E] \sim [Y] - [C],$$

where  $C \subset Y$  is a smooth subvariety,  $\widehat{Y} \rightarrow Y$  is a blow-up at  $C$  and  $E \subset \widehat{Y}$  is the exceptional divisor.

Several interesting invariants of varieties can be extended to invariants of elements in  $K(\text{Var}/\mathbb{C})$ , using the above description of the generators and relations. For instance for a smooth projective variety  $Y$ , its Poincaré polynomial is defined by

$$(32) \quad P_t(Y) = \sum_{i=0}^{2 \dim Y} (-1)^i \dim H^i(Y, \mathbb{C}) t^i.$$

The polynomial  $P_t(*)$  is compatible with respect to the relation (31), hence there is a map,

$$(33) \quad P_t: K(\text{Var}/\mathbb{C}) \rightarrow \mathbb{Z}[t],$$

such that  $P_t([Y])$  coincides with (32) if  $Y$  is smooth and projective.

**4.2. Grothendieck groups of stacks.** The notion of Grothendieck group of varieties can be generalized to that of Artin stacks. For the introduction to stack, the readers can consult [25].

Let  $\mathcal{S}$  be an Artin stack, locally of finite type over  $\mathbb{C}$ . We define the  $\mathbb{Q}$ -vector space  $K(\text{St}/\mathcal{S})$  to be generated by isomorphism classes of symbols

$$[\rho: \mathcal{Y} \rightarrow \mathcal{S}],$$

where  $\mathcal{Y}$  is an Artin stack of finite type over  $\mathbb{C}$ ,  $\rho$  is a 1-morphism, and two symbols  $[\rho_i: \mathcal{Y}_i \rightarrow \mathcal{S}]$  for  $i = 1, 2$  are *isomorphic* if there is a 1-isomorphism of stacks  $f: \mathcal{Y}_1 \xrightarrow{\sim} \mathcal{Y}_2$  with a 2-isomorphism  $\rho_2 \circ f \cong \rho_1$ . For a technical reason, we assume that  $\mathcal{Y}$  has affine geometric stabilizers, i.e. for any  $\mathbb{C}$ -valued point  $y \in \mathcal{Y}(\mathbb{C})$ , the automorphism



group  $\text{Aut}(k(y))$  is an affine algebraic group. The relation is generated by

$$[\rho: \mathcal{Y} \rightarrow \mathcal{S}] \sim [\rho|_{\mathcal{V}}: \mathcal{V} \rightarrow \mathcal{S}] + [\rho|_{\mathcal{U}}: \mathcal{U} \rightarrow \mathcal{S}],$$

where  $\mathcal{V} \subset \mathcal{Y}$  is a closed substack and  $\mathcal{U} := \mathcal{Y} \setminus \mathcal{V}$ .

Let  $P_t$  be the map defined in Lemma 4.1. The following result is proved in [18, Theorem 4.10].

**Lemma 4.1.** *There is a map*

$$P_t: K(\text{St}/\mathcal{S}) \rightarrow \mathbb{Q}(t),$$

such that we have

$$P_t([\rho: [Y/\text{GL}_m(\mathbb{C})] \rightarrow \mathcal{S}]) = \frac{P_t([Y])}{P_t([\text{GL}_m(\mathbb{C})])}.$$

Here  $Y$  is a quasi-projective variety on which  $\text{GL}_m(\mathbb{C})$  acts.

*Proof.* We sketch an outline of the proof. By the assumption that  $\mathcal{Y}$  has affine geometric stabilizers, we can apply Kresch's result [24, Proposition 3.5.9] to show that any element  $u \in K(\text{St}/\mathcal{S})$  is written as a finite sum

$$(34) \quad \sum_{i=1}^k [\rho_i: [Y_i/\text{GL}_{m_i}(\mathbb{C})] \rightarrow \mathcal{S}],$$

where  $Y_i$  is a quasi-projective variety on which  $\text{GL}_{m_i}(\mathbb{C})$  acts. Then we set  $P_t(u)$  to be

$$P_t(u) = \sum_{i=1}^k \frac{P_t([Y_i])}{P_t([\text{GL}_{m_i}(\mathbb{C})])}.$$

The proof given in [18, Theorem 4.10] shows that  $P_t(u)$  does not depend on the expression (34).  $\square$

**Remark 4.2.** *More precisely it is proved in [18, Theorem 4.10] that the map  $P_t$  in Lemma 4.1 satisfies*

$$P_t([\rho: [Y/G] \rightarrow \mathcal{S}]) = \frac{P_t([Y])}{P_t([G])}.$$

Here  $Y$  is a quasi-projective variety and  $G$  is a special algebraic group acting on  $Y$ , where an algebraic group  $G$  is called special if any principal  $G$ -bundle is Zariski-locally trivial. For instance  $\text{GL}_m(\mathbb{C})$ ,  $(\mathbb{C}^*)^k$  are special algebraic groups.

On the other hand, the finite group  $\mathbb{Z}/k\mathbb{Z}$  is not special as  $\mathbb{C}^* \ni z \mapsto z^k \in \mathbb{C}^*$  is not Zariski locally trivial. For instance, let us consider an element of the form  $[\rho: [\text{Spec } \mathbb{C}/G] \rightarrow \mathcal{S}]$  for  $G = \mathbb{Z}/k\mathbb{Z}$ . Then we have

$$[\text{Spec } \mathbb{C}/G] \cong [\mathbb{C}^*/\mathbb{C}^*],$$

where  $\mathbb{C}^*$  acts on  $\mathbb{C}^*$  by  $g \cdot z = g^k z$ . Therefore we have

$$\begin{aligned} P_t([\mathrm{Spec} \mathbb{C}/G] \xrightarrow{\rho} \mathcal{S}) &= \frac{P_t(\mathbb{C}^*)}{P_t(\mathbb{C}^*)} \\ &= 1. \end{aligned}$$

We will need the notions of push-forward and pull-back for the groups  $K(\mathrm{St}/\mathcal{S})$ . Let  $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$  be a morphism of stacks. Then we have the push-forward,

$$f_*: K(\mathrm{St}/\mathcal{S}_1) \rightarrow K(\mathrm{St}/\mathcal{S}_2),$$

defined by

$$f_*[\rho: \mathcal{Y} \rightarrow \mathcal{S}_1] = [f \circ \rho: \mathcal{Y} \rightarrow \mathcal{S}_2].$$

Moreover if  $f$  is of finite type, then we have the pull-back,

$$f^*: K(\mathrm{St}/\mathcal{S}_2) \rightarrow K(\mathrm{St}/\mathcal{S}_1),$$

defined by

$$f^*[\rho: \mathcal{Y} \rightarrow \mathcal{S}_2] = [f^*\rho: \mathcal{Y} \times_{\mathcal{S}_2} \mathcal{S}_1 \rightarrow \mathcal{S}_1].$$

**4.3. Hall algebras of coherent sheaves.** For a smooth projective variety  $X$  over  $\mathbb{C}$ , we denote by  $\mathcal{M}$  the moduli stack of coherent sheaves on  $X$ . Namely  $\mathcal{M}$  is a 2-functor,

$$(35) \quad \mathcal{M}: (\mathrm{Sch}/\mathbb{C}) \rightarrow (\mathrm{groupoid}),$$

which sends a  $\mathbb{C}$ -scheme  $S$  to the groupoid whose objects consist of flat families of coherent sheaves over  $S$ ,

$$\mathcal{E} \in \mathrm{Coh}(X \times S).$$

It is well-known that  $\mathcal{M}$  is an Artin stack which is locally of finite type over  $\mathbb{C}$ .

**Definition 4.3.** We define the  $\mathbb{Q}$ -vector space  $H(X)$  to be

$$H(X) := K(\mathrm{St}/\mathcal{M}).$$

We introduce the  $*$ -product on the  $\mathbb{Q}$ -vector space  $H(X)$ . Let  $\mathcal{E}x$  be the 2-functor,

$$\mathcal{E}x: (\mathrm{Sch}/\mathbb{C}) \rightarrow (\mathrm{groupoid}),$$

which sends a  $\mathbb{C}$ -scheme  $S$  to the groupoid whose objects consist of exact sequences in  $\mathrm{Coh}(X \times S)$ ,

$$(36) \quad 0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0,$$

such that each  $\mathcal{E}_i$  is flat over  $S$ . The stack  $\mathcal{E}x$  is also an Artin stack locally of finite type over  $\mathbb{C}$ . There are 1-morphisms,

$$p_i: \mathcal{E}x \rightarrow \mathcal{M}, \quad i = 1, 2, 3,$$

which send an exact sequence (36) to the object  $\mathcal{E}_i$ . In particular we have the diagram,

$$\begin{array}{ccc} \mathcal{E}x & \xrightarrow{p_2} & \mathcal{M}, \\ (p_1, p_3) \downarrow & & \\ \mathcal{M} \times \mathcal{M}. & & \end{array}$$

Also we define the map

$$\iota: H(X) \otimes H(X) \rightarrow K(\text{St}/\mathcal{M} \times \mathcal{M}),$$

as follows:

$$\iota([\mathcal{Y}_1 \xrightarrow{\rho_1} \mathcal{M}] \otimes [\mathcal{Y}_2 \xrightarrow{\rho_2} \mathcal{M}]) = [\mathcal{Y}_1 \times \mathcal{Y}_2 \xrightarrow{\rho_1 \times \rho_2} \mathcal{M} \times \mathcal{M}].$$

We define  $*$ -product on  $H(X)$  to be

$$(37) \quad * = p_{2*}(p_1, p_3)^* \iota: H(X) \otimes H(X) \rightarrow H(X).$$

The following result is proved in [17].

**Theorem 4.4.** [17, Theorem 5.2] *( $H(X), *$ ) is an associative algebra with unit given by  $\delta_0 = [\text{Spec } \mathbb{C} \xrightarrow{\rho} \mathcal{M}]$ . Here  $\rho(\cdot) = 0 \in \text{Coh}(X)$ .*

Let us look at the  $*$ -product for ‘delta-functions’, corresponding to objects  $E_1, E_2 \in \text{Coh}(X)$ . Namely for an object  $E \in \text{Coh}(X)$ , we set

$$\delta_E = [\rho_E: \text{Spec } \mathbb{C} \rightarrow \mathcal{M}], \quad \rho_E(\cdot) = E.$$

The  $*$ -product  $\delta_{E_1} * \delta_{E_2}$  can be written as

$$(38) \quad \delta_{E_1} * \delta_{E_2} = \left[ \rho: \left[ \frac{\text{Ext}^1(E_2, E_1)}{\text{Hom}(E_2, E_1)} \right] \rightarrow \mathcal{M} \right].$$

Here  $\rho$  is a map sending an element  $u \in \text{Ext}^1(E_2, E_1)$  to the object  $E_3 \in \text{Coh}(X)$ , which fits into the exact sequence,

$$(39) \quad 0 \rightarrow E_1 \rightarrow E_3 \rightarrow E_2 \rightarrow 0,$$

with extension class  $u$ . The vector space  $\text{Hom}(E_2, E_1)$  acts on  $\text{Ext}^1(E_2, E_1)$  trivially. In fact the  $\mathbb{C}$ -valued points of the fiber product

$$(40) \quad (\mathcal{M} \times \mathcal{M}) \times_{(\rho_{E_1} \times \rho_{E_2})} \text{Spec } \mathbb{C},$$

bijectionally correspond to the exact sequences (39), hence elements in  $\text{Ext}^1(E_2, E_1)$ . Given such an extension, the group of the automorphisms of the stack (40) at the  $\mathbb{C}$ -valued point (39) is the kernel of the natural map,

$$\text{Aut}(0 \rightarrow E_1 \rightarrow E_3 \rightarrow E_2 \rightarrow 0) \rightarrow \text{Aut}(E_1) \times \text{Aut}(E_2),$$

which is isomorphic to  $\text{Hom}(E_2, E_1)$ . Hence we have the description (38).

**4.4. Semistable one or zero dimensional sheaves.** In this subsection, we assume that  $X$  is a smooth projective Calabi-Yau 3-fold over  $\mathbb{C}$ . Let  $\omega$  be an  $\mathbb{R}$ -ample divisor on  $X$ . Recall that we constructed a stability condition  $Z_\omega$  on the subcategory

$$\mathrm{Coh}_{\leq 1}(X) \subset \mathrm{Coh}(X),$$

in Example 2.3. Given an element  $(n, \beta) \in \mathbb{Z} \oplus H_2(X, \mathbb{Z})$ , we have the substack,

$$(41) \quad \mathcal{M}_{n, \beta}(\omega) \subset \mathcal{M},$$

which parameterizes  $Z_\omega$ -semistable  $E \in \mathrm{Coh}_{\leq 1}(X)$  satisfying

$$(42) \quad (\chi(E), [E]) = (n, \beta).$$

The substack (41) is known to be an open substack of  $\mathcal{M}$ , which is of finite type over  $\mathbb{C}$ . Furthermore suppose that  $\beta$  and  $n$  are coprime and  $\omega$  is in a general position in the ample cone. Then any  $Z_\omega$ -semistable object  $E \in \mathrm{Coh}_{\leq 1}(X)$  satisfying (42) is  $Z_\omega$ -stable, and the stack  $\mathcal{M}_{n, \beta}(\omega)$  is a  $\mathbb{C}^*$ -gerbe over a projective scheme  $M_{n, \beta}(\omega)$ , i.e.

$$(43) \quad \mathcal{M}_{n, \beta}(\omega) \cong [M_{n, \beta}(\omega)/\mathbb{C}^*].$$

Here  $\mathbb{C}^*$ -acts on  $M_{n, \beta}(\omega)$  trivially. The substack (41) defines the element of  $H(X)$ ,

$$\delta_{n, \beta}(\omega) = [\mathcal{M}_{n, \beta}(\omega) \hookrightarrow \mathcal{M}] \in H(X).$$

Recall that we constructed a map,

$$P_t: H(X) \rightarrow \mathbb{Q}(t),$$

in Lemma 4.1. Applying  $P_t$  to  $\delta_{n, \beta}(\omega)$ , we obtain the element

$$P_t(\delta_{n, \beta}(\omega)) \in \mathbb{Q}(t),$$

which is interpreted as a ‘Poincaré polynomial’ of the moduli stack  $\mathcal{M}_{n, \beta}(\omega)$ .

Suppose that  $\mathcal{M}_{n, \beta}(\omega)$  is written as (43). Then we have

$$(44) \quad \begin{aligned} (t^2 - 1)P_t(\delta_{n, \beta}(\omega)) &= P_t(\mathbb{C}^*)P_t(\delta_{n, \beta}(\omega)) \\ &= P_t(M_{n, \beta}(\omega)). \end{aligned}$$

Hence we can substitute  $t = 1$  to (44) and obtain,

$$(45) \quad \lim_{t \rightarrow 1} (t^2 - 1)P_t(\delta_{n, \beta}(\omega)) = \chi(M_{n, \beta}(\omega)).$$

However when  $n$  and  $\beta$  are not coprime, then  $\mathcal{M}_{n, \beta}(\omega)$  is not necessary written as (43). In this case, as the following example indicates, the rational function (44) may have a pole at  $t = 1$  so the limit (45) does not make sense.

**Example 4.5.** Let  $C \cong \mathbb{P}^1 \subset X$  be a super rigid rational curve as in Example 3.3. Then we have

$$\mathcal{M}_{0,k[C]}(\omega) \cong [\mathrm{Spec} \mathbb{C} / \mathrm{GL}_k(\mathbb{C})],$$

whose closed point correspond to  $\mathcal{O}_C(-1)^{\oplus k}$ . Therefore using [18, Lemma 4.6], we have

$$\begin{aligned} (t^2 - 1)P_t(\delta_{0,k[C]}(\omega)) &= (t^2 - 1) \frac{1}{P_t(\mathrm{GL}_k(\mathbb{C}))} \\ &= \frac{1}{t^{k^2-k} (t^2 - 1) \cdots (t^{2k} - 1)}, \end{aligned}$$

and the limit  $t \rightarrow 1$  does not exist when  $k \geq 2$ .

Instead, we take the ‘logarithm’ of  $\delta_{n,\beta}(\omega)$  in  $H(X)$ .

**Definition 4.6.** We define  $\epsilon_{n,\beta}(\omega) \in H(X)$  to be

$$(46) \quad \epsilon_{n,\beta}(\omega) = \sum_{\substack{l \geq 1, n_i \in \mathbb{Z}, \beta_i \in H_2(X, \mathbb{Z}), 1 \leq i \leq l, \\ n_1 + \cdots + n_l = n, \beta_1 + \cdots + \beta_l = \beta, \\ \arg Z_\omega(n_i, \beta_i) = \arg Z_\omega(n, \beta)}} \frac{(-1)^{l-1}}{l} \delta_{n_1, \beta_1}(\omega) * \cdots * \delta_{n_l, \beta_l}(\omega).$$

Namely for each ray  $l \subset \mathbb{H}$ , if we set

$$\begin{aligned} \delta_l(\omega) &= 1 + \sum_{Z_\omega(n, \beta) \in l} \delta_{n, \beta}(\omega), \\ \epsilon_l(\omega) &= \sum_{Z_\omega(n, \beta) \in l} \epsilon_{n, \beta}(\omega), \end{aligned}$$

then we have

$$\epsilon_l(\omega) = \log \delta_l(\omega).$$

It is shown in [18, Section 6.2] that the function  $(t^2 - 1)P_t(\epsilon_{n,\beta}(\omega))$  has the limit  $t \rightarrow 1$ , hence we obtain the invariant,

$$\widehat{N}_{n,\beta}(\omega) = \lim_{t \rightarrow 1} (t^2 - 1)P_t(\epsilon_{n,\beta}(\omega)) \in \mathbb{Q}.$$

The invariant  $\widehat{N}_{n,\beta}(\omega)$  is interpreted as an ‘Euler characteristic’ of the moduli stack  $\mathcal{M}_{n,\beta}(\omega)$ .

**4.5. Invariants  $N_{n,\beta}$ .** The invariant  $\widehat{N}_{n,\beta}(\omega)$  is interpreted as an unweighted Euler characteristic of  $\mathcal{M}_{n,\beta}(\omega)$ , and we need to involve the Behrend function in order to construct DT type invariants. It is easy to extend the notion of the Behrend function to the locally constructible function on the Artin stack  $\mathcal{M}$ ,

$$\nu_{\mathcal{M}}: \mathcal{M} \rightarrow \mathbb{Z},$$

so that if  $M \rightarrow \mathcal{M}$  is any atlas of relative dimension  $d$ , then  $\nu_{\mathcal{M}} = (-1)^d \nu_M$ . (cf. [19, Proposition 4.4].) We define the map

$$(47) \quad \nu \cdot : H(X) \rightarrow H(X),$$

by sending an element  $[\rho : \mathcal{Y} \rightarrow \mathcal{M}]$  to the element,

$$\sum_{i \in \mathbb{Z}} i[\rho|_{\mathcal{Y}_i} : \mathcal{Y}_i \rightarrow \mathcal{M}],$$

where  $\mathcal{Y}_i = (\nu_{\mathcal{M}} \circ \rho)^{-1}(i)$ .

**Definition 4.7.** We define  $N_{n,\beta}(\omega)$  to be

$$N_{n,\beta}(\omega) = \lim_{t \rightarrow 1} (t^2 - 1) P_t(-\nu \cdot \epsilon_{n,\beta}(\omega)) \in \mathbb{Q}.$$

Again the existence of the limit  $t \rightarrow 1$  is proved in [18, Section 6.2]. A priori, the invariant  $N_{n,\beta}(\omega)$  is defined after we choose a polarization  $\omega$ . However we have the following:

**Lemma 4.8.** The invariant  $N_{n,\beta}(\omega)$  does not depend on a choice of  $\omega$ .

*Proof.* The result is proved in [19, Theorem 6.16].  $\square$

In what follows, we set

$$N_{n,\beta} := N_{n,\beta}(\omega),$$

for some ample divisor  $\omega$  on  $X$ .

**Example 4.9.** (i) Suppose that  $n$  and  $\beta$  are coprime, and  $\omega$  is in a general position. Then  $\mathcal{M}_{n,\beta}(\omega)$  is written as (43) for a projective scheme  $M_{n,\beta}(\omega)$ . Let  $\nu_M$  be the Behrend function on  $M_{n,\beta}(\omega)$ . Then we have

$$\epsilon_{n,\beta}(\omega) = \delta_{n,\beta}(\omega), \quad \nu_{\mathcal{M}}|_{\mathcal{M}_{n,\beta}(\omega)} = -\nu_M,$$

hence we have

$$\begin{aligned} N_{n,\beta} &= \int_{M_{n,\beta}(\omega)} \nu_M d\chi, \\ &= \int_{[M_{n,\beta}(\omega)]^{\text{vir}}} 1. \end{aligned}$$

(ii) In the situation of Example 4.5, we have

$$\delta_{0,[C]}(\omega) = \left[ \frac{\text{Spec } \mathbb{C}}{\mathbb{C}^*} \right], \quad \delta_{0,2[C]}(\omega) = \left[ \frac{\text{Spec } \mathbb{C}}{\text{GL}_2(\mathbb{C})} \right].$$

Therefore we have

$$\begin{aligned} \epsilon_{0,2[C]}(\omega) &= \delta_{0,2[C]}(\omega) - \frac{1}{2} \delta_{0,[C]}(\omega) * \delta_{0,[C]}(\omega) \\ &= \left[ \frac{\text{Spec } \mathbb{C}}{\text{GL}_2(\mathbb{C})} \rightarrow \mathcal{M} \right] - \frac{1}{2} \left[ \frac{\text{Spec } \mathbb{C}}{\mathbb{C}^*} \rightarrow \mathcal{M} \right] * \left[ \frac{\text{Spec } \mathbb{C}}{\mathbb{C}^*} \rightarrow \mathcal{M} \right], \\ &= \left[ \frac{\text{Spec } \mathbb{C}}{\text{GL}_2(\mathbb{C})} \rightarrow \mathcal{M} \right] - \frac{1}{2} \left[ \frac{\text{Spec } \mathbb{C}}{\mathbb{A}^1 \rtimes (\mathbb{C}^*)^2} \rightarrow \mathcal{M} \right]. \end{aligned}$$

The Behrend function  $\nu_{\mathcal{M}}$  is 1 on  $\mathcal{O}_C(-1)^{\oplus 2}$ , hence we have

$$\begin{aligned} & (t^2 - 1)P_t(-\nu \cdot \epsilon_{0,2[C]}(\omega)) \\ &= (t^2 - 1) \left\{ -\frac{1}{t^2(t^2 - 1)(t^4 - 1)} + \frac{1}{2t^2(t^2 - 1)^2} \right\} \\ &= \frac{1}{2t^2(t^2 + 1)}. \end{aligned}$$

By taking the limit  $t \rightarrow 1$ , we obtain  $N_{0,2[C]} = 1/4$ . In general, it can be proved that (cf. [19, Example 6.2])

$$N_{0,k[C]} = \frac{1}{k^2}.$$

(iii) Let us consider the case  $\beta = 0$ . In this case,  $\mathcal{M}_{n,0}(\omega)$  is a moduli stack of length  $n$  zero dimensional sheaves. Explicitly  $\mathcal{M}_{n,0}(\omega)$  is described as follows. Let  $\text{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$  be the Grothendieck Quot scheme which parameterizes quotients

$$(48) \quad \mathcal{O}_X^{\oplus n} \twoheadrightarrow F,$$

with  $F$  zero dimensional length  $n$  sheaves. The group  $\text{GL}_n(\mathbb{C})$  acts on  $\text{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$  via

$$g \cdot (\mathcal{O}_X^{\oplus n} \xrightarrow{s} F) = (\mathcal{O}_X^{\oplus n} \xrightarrow{\text{sg}} F), \quad g \in \text{GL}_n(\mathbb{C}).$$

Let

$$U^{(n)} \subset \text{Quot}^{(n)}(\mathcal{O}_X^{\oplus n}),$$

be the open subscheme corresponding to quotients (48) such that the induced morphism  $H^0(s): \mathbb{C}^{\oplus n} \rightarrow H^0(F)$  is an isomorphism. The  $\text{GL}_n(\mathbb{C})$ -action on  $\text{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$  preserves  $U^{(n)}$ , and the moduli stack  $\mathcal{M}_{n,0}(\omega)$  is written as

$$\mathcal{M}_{n,0}(\omega) \cong [U^{(n)} / \text{GL}_n(\mathbb{C})].$$

In principle, it may be possible to calculate  $N_{n,0}$  using the above description of the moduli stack. (For instance, the computation in [39, Section 5] is applied for  $n = 2$ .) However at this moment, a computation of  $N_{n,0}$  for  $n \geq 3$  is not yet done along with this argument. Instead, we can compute  $N_{n,0}$  using the wall-crossing formula and the computation of  $\text{DT}_0(X)$  in Example 3.5 (i). The result is given in [19, Paragraph 6.3], [23, Paragraph 6.4], [37, Remark 5.14],

$$(49) \quad N_{n,0} = -\chi(X) \sum_{k|n, k \geq 1} \frac{1}{k^2}$$

## 5. WALL-CROSSING IN D0-D2-D6 BOUND STATES

Let  $X$  be a smooth projective Calabi-Yau 3-fold over  $\mathbb{C}$ . In this section, we explain how to deduce the product formula (28) by using the wall-crossing formula. In principle, the result is obtained by combining the arguments in [38], Joyce-Song's wall-crossing formula [19] and the announced result by Behrend-Getzler [6]. However the arguments in [38] are complicated, and we simplify the arguments by using the framework of [37].

**5.1. Category of D0-D2-D6 bound states.** We define the category  $\mathcal{A}_X$  as follows:

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X)[-1] \rangle_{\mathrm{ex}}.$$

In [37, Lemma 3.5], it is proved that  $\mathcal{A}_X$  is the heart of a bounded t-structure on  $\mathcal{D}_X$ ,

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{tr}} \subset D^b \mathrm{Coh}(X),$$

hence in particular  $\mathcal{A}_X$  is an abelian category. The triangulated category  $\mathcal{D}_X$  is called the category of *D0-D2-D6 bound states*.

The heart  $\mathcal{A}_X$  has properties which are required in discussing DT/PT correspondence in Subsection 3.5. For instance if we consider an ideal sheaf  $I_Z$  for a subscheme  $Z \subset X$  with  $\dim Z \leq 1$ , we have the distinguished triangle,

$$(50) \quad \mathcal{O}_Z[-1] \rightarrow I_Z \rightarrow \mathcal{O}_X.$$

Since  $\mathcal{O}_Z[-1]$  and  $\mathcal{O}_X$  are objects in  $\mathcal{A}_X$ , it follows that  $I_Z \in \mathcal{A}_X$  and the sequence (50) is an exact sequence in  $\mathcal{A}_X$ . Also for a stable pair  $(F, s)$ , let  $I^\bullet = (\mathcal{O}_X \xrightarrow{s} F)$  be the associated two term complex with  $\mathcal{O}_X$  located in degree zero and  $F$  in degree one. Then  $I^\bullet$  fits into the distinguished triangle,

$$(51) \quad F[-1] \rightarrow I^\bullet \rightarrow \mathcal{O}_X.$$

By the same argument as above, we have  $I^\bullet \in \mathcal{A}_X$  and the sequence (51) is an exact sequence in  $\mathcal{A}_X$ . As the above argument indicates, the heart  $\mathcal{A}_X$  is expected to be an important category in studying curve counting invariants on Calabi-Yau 3-folds.

**5.2. Comparison with perverse coherent sheaves.** In [1], [36], the notions of *polynomial stability* and *limit stability* are introduced on the following category of *perverse coherent sheaves*,

$$\mathcal{A}^p := \langle \mathrm{Coh}_{\geq 2}(X)[1], \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{ex}}.$$

Here  $\mathrm{Coh}_{\geq 2}(X)$  is the right orthogonal complement of  $\mathrm{Coh}_{\leq 1}(X)$  in  $\mathrm{Coh}(X)$ . In this subsection, we compare  $\mathcal{A}_X$  with  $\mathcal{A}^p$ .

Obviously we have

$$\mathcal{A}_X \subset \mathcal{A}^p[-1].$$



By [36, Lemma 2.16], there exists a torsion pair  $(\mathcal{A}_1^p, \mathcal{A}_{1/2}^p)$  on  $\mathcal{A}^p$ , defined by

$$\begin{aligned}\mathcal{A}_1^p &:= \langle F[1], \mathcal{O}_x : F \text{ is pure two dimensional, } x \in X \rangle_{\text{ex}}, \\ \mathcal{A}_{1/2}^p &:= \{E \in \mathcal{A}^p : \text{Hom}(F, E) = 0 \text{ for any } F \in \mathcal{A}_1^p\}.\end{aligned}$$

Namely we have the following, (cf. [15],)

- For any  $T \in \mathcal{A}_1^p$  and  $F \in \mathcal{A}_{1/2}^p$ , we have  $\text{Hom}(T, F) = 0$ .
- For any  $E \in \mathcal{A}^p$ , there is an exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0,$$

with  $T \in \mathcal{A}_1^p$  and  $F \in \mathcal{A}_{1/2}^p$ .

We set

$$\begin{aligned}(52) \quad \mathcal{A}_{X,1} &:= \mathcal{A}_1^p[-1] \cap \mathcal{A}_X, \\ &= \langle \mathcal{O}_x[-1] : x \in X \rangle_{\text{ex}},\end{aligned}$$

and

$$\begin{aligned}(53) \quad \mathcal{A}_{X,1/2} &:= \mathcal{A}_{1/2}^p[-1] \cap \mathcal{A}_X \\ &= \{E \in \mathcal{A}_X : \text{Hom}(\mathcal{A}_{X,1}, E) = 0\}.\end{aligned}$$

It is easy to check that  $(\mathcal{A}_{X,1}, \mathcal{A}_{X,1/2})$  is a torsion pair on  $\mathcal{A}_X$ , using the fact that  $\mathcal{A}_X$  is noetherian. (cf. [37, Lemma 6.2].) We have the following lemma.

**Lemma 5.1.** *For an object  $E \in \mathcal{A}_{1/2}^p[-1]$ , suppose that*

$$\text{rank}(E) \in \{0, 1\}, \quad c_1(E) = 0.$$

*Then we have  $E \in \mathcal{A}_{X,1/2}$ .*

*Proof.* We only prove the case of  $\text{rank}(E) = 1$ . Take  $E \in \mathcal{A}_{1/2}^p[-1]$  with  $\text{rank}(E) = 1$  and  $c_1(E) = 0$ . Then by [36, Lemma 3.2], we have the exact sequence in  $\mathcal{A}^p[-1]$ ,

$$I_C \rightarrow E \rightarrow F[-1],$$

for some curve  $C \subset X$  and  $F \in \text{Coh}_{\leq 1}(X)$ . Since  $I_C, F[-1] \in \mathcal{A}_X$ , we have  $E \in \mathcal{A}_X$ , hence  $E \in \mathcal{A}_{X,1/2}$ .  $\square$

Below we use the following notation. For  $E, F \in \mathcal{A}_{1/2}^p$ , a morphism  $u: E \rightarrow F$  in  $\mathcal{A}^p$  is called a *strict monomorphism* if  $u$  is injective in  $\mathcal{A}^p$  and  $\text{Cok}(u) \in \mathcal{A}_{1/2}^p$ . Similarly  $u$  is called a *strict epimorphism* if  $u$  is surjective in  $\mathcal{A}^p$  and  $\ker(u) \in \mathcal{A}_{1/2}^p$ . By replacing  $(\mathcal{A}_i^p, \mathcal{A}^p)$  by  $(\mathcal{A}_{X,i}, \mathcal{A}_X)$ , we have the notions of strict monomorphism, strict epimorphism on  $\mathcal{A}_{X,i}$ .

**5.3. Weak stability conditions on  $\mathcal{A}_X$ .** In this subsection, we construct weak stability conditions on  $\mathcal{A}_X$ . (cf. Definition 2.3.) The finitely generated free abelian group  $\Gamma$  is defined by

$$\begin{aligned}\Gamma &:= \mathbb{Z} \oplus H_2(X, \mathbb{Z}) \oplus \mathbb{Z}, \\ &= \Gamma_0 \oplus \mathbb{Z},\end{aligned}$$

where  $\Gamma_0$  is introduced in Example 2.3 (iii). Below we write an element in  $\Gamma$  as  $(n, \beta, r)$  for  $n \in \mathbb{Z}$ ,  $\beta \in H_2(X, \mathbb{Z})$  and  $r \in \mathbb{Z}$ . For an object  $E \in \mathcal{A}_X$ , note that

$$(54) \quad \text{ch}_i(E) \in H^{2i}(X, \mathbb{Z}),$$

since (54) is true for the generating set of objects  $\mathcal{O}_X$  and  $E \in \text{Coh}_{\leq 1}(X)[-1]$ . Therefore the Chern characters define the group homomorphism,

$$\text{cl}: K(\mathcal{A}_X) \rightarrow \Gamma,$$

given by

$$\text{cl}(E) = (\text{ch}_3(E), \text{ch}_2(E), \text{ch}_0(E)).$$

Here we have identified  $H^0(X, \mathbb{Z})$  and  $H^6(X, \mathbb{Z})$  with  $\mathbb{Z}$ , and  $H^2(X, \mathbb{Z})$  with  $H_2(X, \mathbb{Z})$  via Poincaré duality. We take the following 2-step filtration in  $\Gamma$ ,

$$0 = \Gamma_{-1} \subsetneq \Gamma_0 \subsetneq \Gamma_1 = \Gamma,$$

where  $\Gamma_0$  is given in Example 2.3, and the embedding  $\Gamma_0 \hookrightarrow \Gamma$  is given by  $(n, \beta) \mapsto (n, \beta, 0)$ . Hence each subquotient is given by

$$\begin{aligned}\Gamma_0/\Gamma_{-1} &= \mathbb{Z} \oplus H_2(X, \mathbb{Z}), \\ \Gamma_1/\Gamma_0 &= \mathbb{Z}.\end{aligned}$$

Given a following data,

$$(55) \quad \omega \in H^2(X, \mathbb{Q}), \quad 0 < \theta < 1,$$

where  $\omega$  is an ample class, we construct

$$(56) \quad Z_{\omega, \theta} = \{Z_{\omega, \theta, i}\}_{i=0}^1 \in \prod_{i=0}^1 \text{Hom}(\Gamma_i/\Gamma_{i-1}, \mathbb{C}),$$

as follows:

$$\begin{aligned}Z_{\omega, \theta, 0}(n, \beta) &= n - (\omega \cdot \beta)\sqrt{-1}, \\ Z_{\omega, \theta, 1}(r) &= r \exp(i\pi\theta).\end{aligned}$$

Here  $(n, \beta) \in \mathbb{Z} \oplus H_2(X, \mathbb{Z})$  and  $r \in \mathbb{Z}$ . We have the following lemma.

**Lemma 5.2.** *The system of group homomorphisms (56) is a weak stability condition on  $\mathcal{A}_X$ .*

*Proof.* For an object  $E \in \mathcal{A}_X$ , let us take  $i \in \{0, 1\}$  so that  $\text{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$  is satisfied. If  $i = 1$ , then

$$Z_{\omega, \theta}(E) \in \mathbb{R}_{>0} \exp(i\pi\theta) \subset \mathbb{H}.$$

Also if  $i = 0$ , then  $E \in \text{Coh}_{\leq 1}(X)[-1]$  and

$$Z_{\omega, \theta}(E) = Z_{\omega}(E[1]) \in \mathbb{H},$$

where  $Z_{\omega}$  is defined in Example 2.3 (iii). Therefore the condition (i) in Definition 2.4 is satisfied.

We check the condition (ii) in Definition 2.4. Let  $(\mathcal{A}_{X,1}, \mathcal{A}_{X,1/2})$  be the torsion pair of  $\mathcal{A}_X$ , given by (52), (53). For any  $E \in \mathcal{A}_X$ , there is an exact sequence in  $\mathcal{A}_X$ ,

$$(57) \quad 0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0,$$

with  $T \in \mathcal{A}_{X,1}$  and  $F \in \mathcal{A}_{X,1/2}$ . By [36, Lemma 2.19], the categories  $\mathcal{A}_{X,1}$  and  $\mathcal{A}_{X,1/2}$  are finite length (i.e. noetherian and artinian with respect to strict epimorphism and strict monomorphism) quasi-abelian categories. (See [10, Section 4] for the definition of quasi-abelian categories.)

On the other hand, by the same argument of [36, Lemma 2.27], an object  $E \in \mathcal{A}_X$  is  $Z_{\omega, \theta}$ -semistable if and only if one of the following conditions holds:

- We have  $E \in \mathcal{A}_{X,1}$ .
- We have  $E \in \mathcal{A}_{X,1/2}$ , and for any exact sequence

$$0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$$

in  $\mathcal{A}_X$  with  $A, B \in \mathcal{A}_{X,1/2}$ , we have

$$(58) \quad \arg Z_{\omega, \theta}(A) \leq \arg Z_{\omega, \theta}(B).$$

Then for any  $E \in \mathcal{A}_X$ , its Harder-Narasimhan filtration is obtained by combining the sequence (57) and the Harder-Narasimhan filtration of  $F$ , where  $F$  is given by the sequence (57). The existence of the latter filtration is ensured by the fact that  $Z_{\omega, \theta}$ -semistable objects in  $\mathcal{A}_{X,1/2}$  are characterized by the inequality (58) for exact sequences in  $\mathcal{A}_{X,1/2}$ , and  $\mathcal{A}_{X,1/2}$  is of finite length. (See the proof of [36, Theorem 2.29].)  $\square$

We remark that the abelian category  $\mathcal{A}_X$  contains the subcategory,

$$\text{Coh}_{\leq 1}(X)[-1] \subset \mathcal{A}_X,$$

which is closed under subobjects and quotients. Hence for  $F \in \text{Coh}_{\leq 1}(X)$ , the object  $F[-1] \in \mathcal{A}_X$  is  $Z_{\omega, \theta}$ -(semi)stable if and only if  $F$  is  $Z_{\omega}$ -(semi)stable in the sense of Example 2.3 (iii).

Let

$$\text{Stab}_{\Gamma_{\bullet}}(\mathcal{D}_X)$$

be the space of weak stability conditions on  $\mathcal{D}_X$ , as in (7). It is straightforward to check that the pairs  $(Z_{\omega,\theta}, \mathcal{A}_X)$  satisfy the conditions required to construct the space  $\text{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ , i.e. local finiteness, support property in [37, Section 2]. Therefore by applying [37, Lemma 7.1], we have the continuous morphism for a fixed  $\omega$ ,

$$(59) \quad (0, 1) \ni \theta \mapsto (Z_{\omega,\theta}, \mathcal{A}_X) \in \text{Stab}_{\Gamma_\bullet}(\mathcal{D}_X).$$

**5.4. Comparison with  $\mu$ -limit stability.** Let us take

$$B + i\omega \in H^2(X, \mathbb{C})$$

with  $\omega$  ample. Below we set  $B = k\omega$  for  $k \in \mathbb{R}$ . In [38], the author introduced the notion of  $\mu_{B+i\omega}$ -limit stability on the abelian category  $\mathcal{A}^p$ . Suppose that an object  $E \in \mathcal{A}^p[-1]$  satisfies

$$(60) \quad \text{ch}(E) = (1, 0, -\beta, -n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6,$$

Then by [38, Lemma 3.8] and [38, Proposition 3.13], an object  $E[1] \in \mathcal{A}^p$  is  $\mu_{B+i\omega}$ -limit semistable if and only if  $E \in \mathcal{A}_{1/2}^p$  and the following conditions are satisfied:

- For any pure one dimensional sheaf  $0 \neq F$  which admits a strict monomorphism  $F \hookrightarrow E[1]$  in  $\mathcal{A}_{1/2}^p$ , we have  $\text{ch}_3(F)/\omega \text{ch}_2(F) \leq -2k$ .
- For any pure one dimensional sheaf  $0 \neq G$  which admits a strict epimorphism  $E[1] \twoheadrightarrow G$  in  $\mathcal{A}_{1/2}^p$ , we have  $\text{ch}_3(G)/\omega \text{ch}_2(G) \geq -2k$ .

Now we set

$$(61) \quad k = \frac{1}{2 \tan \pi \theta}.$$

Here  $k = 0$  if  $\theta = 1/2$ . By Lemma 5.1 and the arguments in the proof of Lemma 5.2, the following lemma obviously follows.

**Lemma 5.3.** *Take  $k$  and  $\theta$  satisfying (61). Then for an object  $E \in \mathcal{A}^p[-1]$  satisfying (60),  $E[1] \in \mathcal{A}^p$  is  $\mu_{k\omega+i\omega}$ -limit semistable in the sense of [38, Section 3] if and only if  $E \in \mathcal{A}_X$  and  $E$  is  $Z_{\omega,\theta}$ -semistable satisfying*

$$\text{cl}(E) = (-n, -\beta, 1) \in \Gamma.$$

**5.5. Moduli stacks of semistable objects.** In this subsection, we discuss moduli stacks of semistable objects in  $\mathcal{A}_X$ . We denote by  $\widehat{\mathcal{M}}$  the 2-functor,

$$\widehat{\mathcal{M}}: (\text{Sch}/\mathbb{C}) \rightarrow (\text{groupoid}),$$

which sends a  $\mathbb{C}$ -scheme  $S$  to the groupoid whose objects consist of objects

$$\mathcal{E} \in D(\text{Coh}(X \times S)),$$

such that

- The object  $\mathcal{E}$  is relatively perfect over  $S$ . (See [29, Definition 2.1.1].) In particular for each  $s \in S$ , we have the derived pull-back,

$$(62) \quad \mathcal{E}_s := \mathbf{L}i_s^* \mathcal{E} \in D^b \text{Coh}(X).$$

Here  $i_s: X \times \{s\} \hookrightarrow X \times S$  is the inclusion.

- The object (62) satisfies

$$\text{Ext}^i(\mathcal{E}_s, \mathcal{E}_s) = 0, \quad i < 0,$$

for any  $s \in S$ .

By the result of Lieblich [29], the 2-functor  $\widehat{\mathcal{M}}$  is an Artin stack locally of finite type over  $\mathbb{C}$ . We note that the stack  $\mathcal{M}$  considered in (35) is an open substack of  $\widehat{\mathcal{M}}$ .

Let  $\mathcal{O}bj(\mathcal{A}_X)$  be the (abstract) substack,

$$\mathcal{O}bj(\mathcal{A}_X) \subset \widehat{\mathcal{M}},$$

whose  $S$ -valued points consist of  $\mathcal{E} \in \widehat{\mathcal{M}}(S)$  satisfying  $\mathcal{E}_s \in \mathcal{A}_X$  for all  $s \in S$ . The stack  $\mathcal{O}bj(\mathcal{A}_X)$  decomposes as

$$\mathcal{O}bj(\mathcal{A}_X) = \coprod_{v \in \Gamma} \mathcal{O}bj^v(\mathcal{A}_X),$$

where  $\mathcal{O}bj^v(\mathcal{A}_X)$  is the stack of objects  $E \in \mathcal{A}_X$  with  $\text{cl}(E) = v$ . As proved in [37, Lemma 3.16], the embedding

$$\mathcal{O}bj^v(\mathcal{A}_X) \subset \widehat{\mathcal{M}},$$

is an open immersion if  $v = (n, \beta, r) \in \Gamma$  with  $r = 0$  or  $r = 1$ . In particular in that case,  $\mathcal{O}bj^v(\mathcal{A}_X)$  is an Artin stack locally of finite type over  $\mathbb{C}$ . In general,  $\mathcal{O}bj^v(\mathcal{A}_X)$  is at least a locally constructible subset of  $\widehat{\mathcal{M}}$ .

Let  $\omega$  and  $\theta$  be as in (55). We define

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \subset \mathcal{O}bj^{(-n, -\beta, 1)}(\mathcal{A}_X),$$

to be the stack which parameterizes  $Z_{\omega, \theta}$ -semistable objects  $E \in \mathcal{A}_X$  with  $\text{cl}(E) = (-n, -\beta, 1)$ . We have the following proposition.

**Proposition 5.4.** (i) *The stack  $\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta)$  is an Artin stack of finite type over  $\mathbb{C}$ .*

(ii) *If  $\theta$  is sufficiently close to 1, then we have*

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \cong [P_n(X, \beta)/\mathbb{G}_m],$$

where  $\mathbb{G}_m$  acts on  $P_n(X, \beta)$  trivially.

(iii) *We have the isomorphism,*

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \xrightarrow{\cong} \widehat{\mathcal{M}}_{-n,\beta}(\omega, 1 - \theta),$$

given by

$$E \mapsto \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X).$$

(iv) We have

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta = 1/2) = \emptyset,$$

for  $|n| \gg 0$ .

*Proof.* By Lemma 5.3, the stack  $\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta)$  is identified with the moduli stack of  $\mu_{k\omega+i\omega}$ -limit semistable objects  $E \in \mathcal{A}_{1/2}^p$  satisfying (60), where  $k$  is given by (61). The results of the proposition follow from the corresponding results for  $\mu_{k\omega+i\omega}$ -limit stability. Namely (i) follows from [38, Proposition 3.17], (ii) follows from [38, Theorem 3.21], (iii) follows from [36, Lemma 2.28] and (iv) follows from [38, Lemma 4.4].  $\square$

**5.6. Rank one counting invariants.** Using the moduli stack  $\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta)$ , we are able to construct the invariant,

$$\mathrm{DT}_{n,\beta}(\omega, \theta) \in \mathbb{Q},$$

which counts  $Z_{\omega,\theta}$ -semistable  $E \in \mathcal{A}_X$  with  $\mathrm{cl}(E) = (-n, -\beta, 1)$ . Namely, suppose that any  $Z_{\omega,\theta}$ -semistable object  $E \in \mathcal{A}_X$  with  $\mathrm{cl}(E) = (-n, -\beta, 1)$  is  $Z_{\omega,\theta}$ -stable. (This is true if  $\omega$  and  $\theta$  are chosen to be generic.) Then we have

$$(63) \quad \widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \cong [\widehat{M}_{n,\beta}(\omega, \theta)/\mathbb{G}_m]$$

for an algebraic space  $\widehat{M}_{n,\beta}(\omega, \theta)$  of finite type over  $\mathbb{C}$ . If  $\nu_M$  is the Behrend function on  $\widehat{M}_{n,\beta}(\omega, \theta)$ , then we can define

$$\mathrm{DT}_{n,\beta}(\omega, \theta) = \int_{\widehat{M}_{n,\beta}(\omega, \theta)} \nu_M d\chi.$$

On the other hand, suppose that there is a strictly  $Z_{\omega,\theta}$ -semistable object  $E \in \mathcal{A}_X$  satisfying  $\mathrm{cl}(E) = (-n, -\beta, 1)$ . Then the stack  $\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta)$  is not written in a way (63), and we need to modify the definition of  $\mathrm{DT}_{n,\beta}(\omega, \theta)$  using the Hall type algebra as we discussed in the previous section. Namely we consider

$$H(\mathcal{A}_X) := K_0(\mathrm{St}/\mathrm{Obj}(\mathcal{A}_X)),$$

and the  $*$ -product on  $H(\mathcal{A}_X)$  given in a similar way to (37), by replacing  $\widehat{\mathcal{M}}$  by  $\mathrm{Obj}(\mathcal{A}_X)$ . By Proposition 5.4, we can define the elements in  $\mathcal{H}(\mathcal{A}_X)$ ,

$$\begin{aligned} \widehat{\delta}_{n,\beta}(\omega) &= [\mathcal{M}_{n,\beta}(\omega) \xrightarrow{i} \mathrm{Obj}(\mathcal{A}_X)], \\ \widehat{\delta}_{n,\beta}(\omega, \theta) &= [\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \hookrightarrow \mathrm{Obj}(\mathcal{A}_X)], \end{aligned}$$

where  $\mathcal{M}_{n,\beta}(\omega)$  is the stack introduced in (41), and  $i$  sends  $E \in \text{Coh}_{\leq 1}(X)$  to  $E[-1] \in \mathcal{A}_X$ . Its ‘logarithm’ is defined by,

$$\begin{aligned} \widehat{\epsilon}_{n,\beta}(\omega, \theta) = & \sum_{\substack{l \geq 1, 1 \leq e \leq l, (n_i, \beta_i) \in \mathbb{Z} \oplus H_2(X, \mathbb{Z}), \\ n_1 + \dots + n_l = n, \beta_1 + \dots + \beta_l = \beta \\ Z_{\omega, \theta}(-n_i, -\beta_i, 0) \in \mathbb{R}_{>0} \exp(i\pi\theta), i \neq e.}} \frac{(-1)^{l-1}}{l} \widehat{\delta}_{n_1, \beta_1}(\omega) * \dots * \widehat{\delta}_{n_{e-1}, \beta_{e-1}}(\omega) \\ & * \widehat{\delta}_{n_e, \beta_e}(\omega, \theta) * \widehat{\delta}_{n_{e+1}, \beta_{e+1}}(\omega) * \dots * \widehat{\delta}_{n_l, \beta_l}(\omega). \end{aligned}$$

Then  $\text{DT}_{n,\beta}(\omega, \theta) \in \mathbb{Q}$  can be defined by

$$\text{DT}_{n,\beta}(\omega, \theta) = \lim_{t \rightarrow 1} (t^2 - 1) P_t(-\nu \cdot \epsilon_{n,\beta}(\omega, \theta)),$$

where  $\nu$  is defined similarly to (47) by using the Behrend function on  $\mathcal{O}bj(\mathcal{A}_X)$ . Also see [38, Definition 4.1], [37, Definition 4.11]. We define the invariant  $L_{n,\beta} \in \mathbb{Q}$  as follows.

**Definition 5.5.** *We define  $L_{n,\beta} \in \mathbb{Q}$  to be*

$$L_{n,\beta} := \text{DT}_{n,\beta}(\omega, \theta = 1/2).$$

As a corollary of Proposition 5.4, we have the following:

**Corollary 5.6.** *(i) If  $\theta$  is sufficiently close to 1, we have*

$$\text{DT}_{n,\beta}(\omega, \theta) = P_{n,\beta}.$$

*(ii) The invariant  $L_{n,\beta}$  satisfies,*

$$L_{n,\beta} = L_{-n,\beta},$$

*and they are zero for  $|n| \gg 0$ .*

**5.7. Wall-crossing formula.** We define the series  $\text{DT}(\omega, \theta)$  by

$$(64) \quad \text{DT}(\omega, \theta) := \sum_{n,\beta} \text{DT}_{n,\beta}(\omega, \theta) q^n t^\beta.$$

Similarly to [37, Definition 4.11], [35, Section 4.3], the series (64) can be defined in a certain topological vector space for  $0 < \theta < 1/2$ . Also as in [37, Subsection 5.1], it is straightforward to check the existence of wall and chamber structure on the space  $\text{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ . Therefore the following limiting series makes sense for  $\phi \in (0, 1/2)$ ,

$$\text{DT}(\omega, \phi_\pm) := \lim_{\theta \rightarrow \phi \pm 0} \text{DT}(\omega, \theta).$$

Using Joyce-Song’s wall-crossing formula [19] and assuming the result by Behrend-Getzler <sup>2</sup> [6], we have the following theorem. (Also see Remark 3.12 and [35, Remark 2.32, Conjecture 4.3].)

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<sup>2</sup>The result of [6] is not yet written at the moment the author writes this manuscript

**Theorem 5.7.** *For  $0 < \phi < 1/2$ , we have the following formula,*

$$(65) \quad \text{DT}(\omega, \phi_+) = \text{DT}(\omega, \phi_-) \cdot \prod_{\substack{n>0, \beta>0 \\ -n+(\omega \cdot \beta)i \in \mathbb{R}_{>0} e^{i\pi\phi}}} \exp((-1)^{n-1} n N_{n,\beta} q^n t^\beta).$$

*Proof.* Let us fix  $\omega$  and consider the subset

$$\mathcal{V} \subset \text{Stab}_{\Gamma_\bullet}(\mathcal{D}_X),$$

defined by the image of the map (59). Then it is easy to check that the subspace  $\mathcal{V}$  satisfies the assumptions of [37, Assumption 4.1]. Therefore the result follows from [37, Theorem 5.8, Theorem 8.10 (arXiv version)].  $\square$

As a corollary of the above theorem, we obtain the desired product expansion (28).

**Corollary 5.8.** *We have the formula,*

$$(66) \quad \text{PT}(X) = \prod_{n>0, \beta>0} \exp((-1)^{n-1} n N_{n,\beta} q^n t^\beta) \left( \sum_{n,\beta} L_{n,\beta} q^n t^\beta \right).$$

*Proof.* By Corollary 5.6, we have

$$\lim_{\theta \rightarrow 1} \text{DT}(\omega, \theta) = \text{PT}(X).$$

On the other hand, note that if  $F \in \text{Coh}_{\leq 1}(X)$  satisfies

$$Z_{\omega, 1/2}(F[-1]) \in \mathbb{R}_{>0} \sqrt{-1},$$

then  $\chi(F) = 0$ . Using this fact and following the argument of [37, Theorem 5.8, Theorem 8.10], it can be checked that

$$\begin{aligned} \lim_{\theta \rightarrow 1/2} \text{DT}(\omega, \theta) &= \text{DT}(\omega, \theta = 1/2), \\ &= \sum_{n,\beta} L_{n,\beta} q^n t^\beta. \end{aligned}$$

Therefore applying wall-crossing formula (65) from  $\theta = 1/2$  to  $\theta \rightarrow 1$ , we obtain the formula (66). (See [37, Corollary 5.11] to justify this argument.)  $\square$

## 6. PRODUCT EXPANSION FORMULA

In this section, we discuss a conjectural product expansion formula of the series  $\text{PT}(X)$ , and see how it is related to our formula (66). It leads to a conjectural multi-covering formula of the invariant  $N_{n,\beta}$ , and we will give its evidence in a specific example.



**6.1. Gopakumar-Vafa formula.** For  $g \geq 0$  and  $\beta \in H_2(X, \mathbb{Z})$ , the GW invariant  $N_{g,\beta}^{\text{GW}} \in \mathbb{Q}$  is not an integer in general. However Gopakumar-Vafa [14] claims the following integrality of  $N_{g,\beta}^{\text{GW}}$ , based on the string duality between Type IIA string theory and M-theory.

**Conjecture 6.1.** *There are integers*

$$n_g^\beta \in \mathbb{Z}, \text{ for } g \geq 0, \beta \in H_2(X, \mathbb{Z}),$$

*such that we have*

$$(67) \quad \sum_{g \geq 0, \beta > 0} N_{g,\beta}^{\text{GW}} \lambda^{2g-2} t^\beta = \sum_{g \geq 0, \beta > 0, k \in \mathbb{Z}_{\geq 1}} \frac{n_g^\beta}{k} \left( 2 \sin \left( \frac{k\lambda}{2} \right)^{2g-2} \right) t^{k\beta}.$$

The invariant  $n_g^\beta \in \mathbb{Z}$  is called a *Gopakumar-Vafa invariant*. The LHS of (67) can be always written as in the RHS of (67) for some  $n_g^\beta \in \mathbb{Q}$ , but the integrality of  $n_g^\beta$  is not obvious. The above conjecture is implied by GW/DT/PT correspondence, noting that DT or PT invariants are integers. (cf. [32, Theorem 3.19].)

Now let us believe GW/DT/PT correspondence and write GW generating series in the Gopakumar-Vafa form (67). Then the series  $\text{PT}(X)$  should be written as a certain conjectural formula involving  $n_g^\beta$ . The expected formula is formulated in [21]:

**Conjecture 6.2.** *There are integers*

$$n_g^\beta \in \mathbb{Z}, \text{ for } g \geq 0, \beta \in H_2(X, \mathbb{Z}),$$

*such that we have*

$$(68) \quad \text{PT}(X) = \prod_{\beta > 0} \left( \prod_{j=1}^{\infty} (1 - (-q)^j t^\beta)^{jn_0^\beta} \cdot \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} (1 - (-q)^{g-1-k} t^\beta)^{(-1)^{k+g} n_g^\beta \binom{2g-2}{k}} \right).$$

The above conjecture is nothing but the strong rationality conjecture discussed in [32]. In what follows we discuss the relationship between the formulas (66) and (68).

**6.2. Multi-covering formula of  $N_{n,\beta}$ .** First let us take the logarithm of the RHS of (68). Then we obtain

$$\begin{aligned}
& \log \prod_{\beta>0} \prod_{j=1}^{\infty} (1 - (-q)^j t^\beta)^{jn_0^\beta} \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} (1 - (-q)^{g-1-k} t^\beta)^{(-1)^{k+g} n_g^\beta \binom{2g-2}{k}} \\
(69) \quad & = \sum_{\beta>0} \sum_{j=1}^{\infty} j n_0^\beta \log (1 - (-q)^j t^\beta) \\
& \quad + \sum_{\beta>0} \sum_{g=1}^{\infty} \sum_{k=0}^{2g-2} (-1)^{k+g} n_g^\beta \binom{2g-2}{k} \log (1 - (-q)^{g-1-k} t^\beta) \\
(70) \quad & = \sum_{\beta>0} \sum_{j=1}^{\infty} j n_0^\beta \sum_{k \geq 1} \frac{(-1)^{jk-1} q^{jk}}{k} t^{k\beta} \\
& \quad + \sum_{\beta>0} \sum_{g=1}^{\infty} \sum_{a \geq 1} \frac{n_g^\beta}{a} \sum_{k=0}^{2g-2} \binom{2g-2}{k} \{ -(-q)^a \}^{g-1-k} t^{a\beta}.
\end{aligned}$$

The first term of (70) is written as

$$(71) \quad \sum_{\beta>0} \sum_{n=1}^{\infty} \sum_{k \geq 1, k|(\beta,n)} \frac{(-1)^{n-1} n}{k^2} n_0^{\beta/k} q^n t^\beta,$$

and the coefficient of  $t^\beta$  is an element of  $q\mathbb{Q}[[q]]$ . As for the second term of (70), we set

$$\begin{aligned}
f_g(q) &:= \sum_{k=0}^{2g-2} \binom{2g-2}{k} q^{g-1-k} \\
(72) \quad &= q^{1-g} (1+q)^{2g-2}.
\end{aligned}$$

Then the second term of (70) is written as

$$(73) \quad \sum_{\beta>0} \sum_{g=1}^{\infty} \sum_{a \geq 1, a|\beta} \frac{n_g^{\beta/a}}{a} f_g(-(-q)^a) t^\beta.$$

Note that the coefficient of  $t^\beta$  in (73) is a polynomial of  $q^{\pm 1}$  invariant under  $q \leftrightarrow 1/q$ .

Next taking the logarithm of (66), we obtain

$$(74) \quad \log \text{PT}(X) = \sum_{\beta>0} \sum_{n>0} (-1)^{n-1} n N_{n,\beta} q^n t^\beta + \log \left( \sum_{n,\beta} L_{n,\beta} q^n t^\beta \right).$$

The coefficient of  $t^\beta$  in the first term of the RHS of (74) is an element of  $q\mathbb{Q}[[q]]$ . We set

$$(75) \quad \sum_{\beta > 0} L_\beta(q) t^\beta := \log \left( \sum_{n, \beta} L_{n, \beta} q^n t^\beta \right).$$

Then  $L_\beta(q)$  is a polynomial of  $q^{\pm 1}$  which is invariant under  $q \leftrightarrow 1/q$ .

For a Laurent series  $F(q)$  in  $q$ , note that the decomposition

$$\begin{aligned} F(q) &= F_1(q) + F_2(q), \\ F_1(q) &\in q\mathbb{Q}[[q]], F_2(q) \in \mathbb{C}[q^{\pm 1}], \end{aligned}$$

is unique if  $F_2(q)$  is invariant under  $q \leftrightarrow 1/q$ . Hence if Conjecture 6.2 holds, the comparison of (70) with (74) gives

$$(76) \quad \sum_{n > 0} (-1)^{n-1} n N_{n, \beta} q^n = \sum_{n=1}^{\infty} \sum_{k \geq 1, k | (\beta, n)} \frac{(-1)^{n-1} n}{k^2} n_0^{\beta/k} q^n,$$

$$(77) \quad L_\beta(q) = \sum_{g=1}^{\infty} \sum_{a \geq 1, a | \beta} \frac{n_g^{\beta/a}}{a} f_g(-(-q)^a).$$

By looking at the coefficient of  $q$  in (76), we obtain

$$N_{1, \beta} = n_{0, \beta}.$$

Then by looking at the coefficient of  $q^n$ , we obtain the following conjectural formula.

**Conjecture 6.3.** *We have the following formula,*

$$(78) \quad N_{n, \beta} = \sum_{k \geq 1, k | (n, \beta)} \frac{1}{k^2} N_{1, \beta/k}.$$

By the above argument, if Conjecture 6.3 is true, then  $n_0^\beta = N_{1, \beta}$  satisfies the equation (76). Note that  $N_{1, \beta}$  is an integer since the vector  $(1, \beta)$  is primitive.

Also the equation (77) gives a way to write down  $n_g^\beta$  for  $g \geq 1$  in terms of  $L_{n, \beta}$ . Namely if  $G(q) \in \mathbb{Q}[q^{\pm 1}]$  is invariant under  $q \leftrightarrow 1/q$ , then there is a unique way to write  $G(q)$  as

$$G(q) = \sum_{g=1}^N a_g f_g(q),$$

with  $a_g \in \mathbb{Q}$ . Hence we are able to write down  $n_g^\beta$  in terms  $L_{n, \beta}$  using the equation (77) recursively. For instance, as we will see in Theorem 6.6, we have

$$(79) \quad n_1^\beta = \sum_n (-1)^n L_{n, \beta} - \frac{1}{2} \sum_{n_1, n_2} \sum_{\beta_1 + \beta_2 = \beta} (-1)^{n_1 + n_2} L_{n_1, \beta_1} L_{n_2, \beta_2} + \cdots,$$

if  $\beta$  is a primitive curve class. The integrality of  $n_g^\beta$  for  $g \geq 1$  is not obvious from the expression of  $n_g^\beta$  in terms of  $L_{n,\beta}$ , as in (79). However by [32, Theorem 3.19], if  $\text{PT}(X)$  is once written as a product expansion (68), then the integrality of  $n_g^\beta$  follows from the integrality of  $P_{n,\beta} \in \mathbb{Z}$ . As a summary, we obtain the following.

**Theorem 6.4.** *Conjecture 6.2 is equivalent to Conjecture 6.3. In that case, we have*

$$n_0^\beta = N_{1,\beta},$$

and there is a way to write down  $n_g^\beta$  for  $g \geq 1$  in terms of  $L_{n,\beta}$ .

**Remark 6.5.** *The invariant  $N_{1,\beta}$  is nothing but Katz's definition of genus zero Gopakumar-Vafa invariant [20].*

**6.3. Higher genus Gopakumar-Vafa invariants.** As we observed in Theorem 6.4, if we assume Conjecture 6.2, then  $n_g^\beta$  is written in terms of  $L_{n,\beta}$ . The purpose of this subsection is to give its explicit formula.

For  $m \geq 0$ , we set  $h_m(q)$  by

$$h_m(q) = \begin{cases} 1, & m = 0, \\ q^m + q^{-m}, & m \geq 1. \end{cases}$$

Let  $f_g(q)$  be the function defined by (72). Then for  $g \geq 1$ , we have

$$(80) \quad f_g(q) = \sum_{m=0}^{g-1} \binom{2g-2}{g-1+m} h_m(q).$$

There is an inversion formula of (80). Namely there are  $c_g^{(m)} \in \mathbb{Z}$  such that

$$(81) \quad h_m(q) = \sum_{g=1}^{m+1} c_g^{(m)} f_g(q).$$

An elementary calculation shows that  $c_g^{(m)}$  is given by

$$(82) \quad c_g^{(m)} = (-1)^{m+g-1} \left\{ \binom{m+g}{2g-1} - \binom{m+g-2}{2g-1} \right\}.$$

The Möbius function on  $\mathbb{Z}_{\geq 1}$  is defined as follows:

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \text{if } n \text{ is square free,} \\ 0, & \text{otherwise.} \end{cases}$$

Here  $\omega(n)$  is the number of distinct prime factors of  $n$ . Then by (77) and the Möbius inversion formula, we have

$$(83) \quad \sum_{g \geq 1} n_g^\beta f_g(q) = \sum_{a \geq 1, a|\beta} \frac{\mu(a)}{a} L_{\beta/a}(-(-q)^a).$$

If we write

$$(84) \quad L_\beta(q) = \sum_{n,\beta} L'_{n,\beta} q^n,$$

for  $L'_{n,\beta} \in \mathbb{Q}$ , then we have

$$\begin{aligned} (83) &= \sum_{a \geq 1, a|\beta} \frac{\mu(a)}{a} \sum_{n \in \mathbb{Z}} L'_{n,\beta/a} (-1)^{na+n} q^{na} \\ &= \sum_{a \geq 1, a|\beta} \frac{\mu(a)}{a} \sum_{n \geq 0} (-1)^{na+n} L'_{n,\beta/a} h_{na} \\ &= \sum_{n \geq 0} \sum_{a \geq 1, a|(n,\beta)} \frac{\mu(a)}{a} (-1)^{n+n/a} L'_{n/a,\beta/a} h_n \\ &= \sum_{g \geq 1} \left( \sum_{n \geq g-1} \sum_{a \geq 1, a|(n,\beta)} \frac{\mu(a)}{a} (-1)^{n+n/a} L'_{n/a,\beta/a} c_g^{(n)} \right) f_g(q). \end{aligned}$$

Here we have used (81) for the last equality. On the other hand, comparing (75) with (84), we have

$$L'_{n,\beta} = \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{n_1 + \dots + n_l = n, \\ \beta_1 + \dots + \beta_l = \beta}} \prod_{i=1}^l L_{n_i, \beta_i}.$$

Also using the formula (82) for  $c_g^{(n)}$ , we obtain the following result.

**Theorem 6.6.** *Suppose that Conjecture 6.2 is true. Then  $n_0^\beta = N_{1,\beta}$  and  $n_g^\beta$  for  $g \geq 1$  is given by*

$$n_g^\beta = \sum_{\substack{n \geq g-1, \\ a \geq 1, a|(n,\beta)}} \sum_{\substack{l \geq 1, \\ n_1 + \dots + n_l = n/a, \\ \beta_1 + \dots + \beta_l = \beta/a}} \frac{\mu(a)}{al} (-1)^{l+g+n/a} \left\{ \binom{n+g}{2g-1} - \binom{n+g-2}{2g-1} \right\} \cdot \prod_{i=1}^l L_{n_i, \beta_i}.$$

**6.4. Example: Weierstrass model.** We will prove Conjecture 6.2 and compute  $n_{g,\beta}$  in the following specific example. Let  $S$  be a smooth projective del-Pezzo surface over  $\mathbb{C}$ . Take general elements,

$$f \in \Gamma(S, \mathcal{O}_S(-4K_S)), \quad g \in \Gamma(S, \mathcal{O}_S(-6K_S)).$$

We construct a Calabi-Yau 3-fold with an elliptic fibration,

$$\pi: X \rightarrow S,$$

by the defining equation

$$y^2 = x^3 + fx + g,$$

in the projective bundle,

$$\mathcal{P}rojSym_S^\bullet(\mathcal{O}_S \oplus \mathcal{O}_S(-2K_S) \oplus \mathcal{O}_S(-3K_S)) \rightarrow S.$$

Here  $x$  and  $y$  are local sections of  $\mathcal{O}_S(-2K_S)$  and  $\mathcal{O}_S(-3K_S)$  respectively. A Calabi-Yau 3-fold  $X$  constructed in this way is called a *Weierstrass model*. A general fiber of  $\pi: X \rightarrow S$  is a smooth elliptic curve, and any singular fiber is either a nodal or cuspidal plane curve.

Let  $F \subset X$  be a general fiber of  $\pi$ . We study the following series,

$$\mathrm{PT}(X/S) := \sum_{n,m} \mathrm{PT}_{n,m[F]} q^n t^m.$$

By the formula (66), we have the product expansion formula,

(85)

$$\mathrm{PT}(X/S) = \prod_{n>0, m>0} \exp((-1)^{n-1} n N_{n,m[F]} q^n t^m) \left( \sum_{n,m} L_{n,m[F]} q^n t^m \right).$$

In what follows, we omit  $[F]$  in the notation for simplicity. So for instance, we write  $N_{n,m[F]}$  as  $N_{n,m}$ .

**Proposition 6.7.** *The invariant  $N_{n,m}$  satisfies the formula (78), and*

$$N_{1,m} = -\chi(X).$$

*Proof.* Let  $\omega_X$  be an ample divisor on  $X$ . Let

$$\mathcal{M}_{n,m}^s(\omega_X) \subset \mathcal{M}_{n,m}(\omega_X),$$

be the substack corresponding to  $Z_{\omega_X}$ -stable objects in  $\mathrm{Coh}_{\leq 1}(X)$ , introduced in Example 2.3 (iii). Note that if  $E \in \mathrm{Coh}_{\leq 1}(X)$  represents a closed point of  $\mathcal{M}_{n,m}^s(\omega_X)$ , then  $E$  is written as

$$(86) \quad E \cong i_{p*} E',$$

for some stable sheaf  $E'$  on an elliptic fiber  $\pi^{-1}(p)$  for some  $p \in S$ . Here  $i_p: \pi^{-1}(p) \hookrightarrow X$  is the inclusion. By the classification of stable sheaves on the fibers of  $\pi$  given in [8], we have

$$(87) \quad \mathcal{M}_{n,m}^s(\omega_X) = \emptyset, \quad \text{if } \mathrm{g.c.d.}(n, m) > 1.$$

Assume that  $\mathrm{g.c.d.}(n, m) = 1$ . Let

$$Y \rightarrow S,$$

be the relative moduli space of  $Z_{\omega_X}$ -stable sheaves  $E$  on the fibers of  $\pi: X \rightarrow S$ , satisfying

$$(88) \quad [E] = m[F], \quad \chi(E) = n.$$

By the condition  $\mathrm{g.c.d.}(n, m) = 1$  and the result of [11], the variety  $Y$  is smooth projective, irreducible, and there is a derived equivalence,

$$(89) \quad \Phi: D^b \mathrm{Coh}(X) \xrightarrow{\sim} D^b \mathrm{Coh}(Y),$$

which takes any  $Z_{\omega_X}$ -stable sheaf satisfying (88) to an object of the form  $\mathcal{O}_y$  for a closed point  $y \in Y$ . For  $d \in \mathbb{Z}_{\geq 1}$ , take a  $\mathbb{C}$ -valued point,

$$[E] \in \mathcal{M}_{(dn, dm)}(\omega_X).$$

By (87), any Jordan-Hölder factor of  $E$  determines a closed point in  $\mathcal{M}_{n,m}(\omega_X)$ . Hence the equivalence  $\Phi$  induces the isomorphism,

$$\mathcal{M}_{(dn,dm)}(\omega_X) \xrightarrow{\sim} \mathcal{M}_{(d,0)}(\omega_Y).$$

Here  $\omega_Y$  is an arbitrary polarization on  $Y$ . (Obviously the RHS does not depend on  $\omega_Y$ .) Therefore we obtain that

$$\begin{aligned} N_{dn,dm}(\omega_X) &= N_{d,0}(\omega_Y) \\ &= -\chi(Y) \sum_{k \geq 1, k|d} \frac{1}{k^2}, \\ &= -\chi(X) \sum_{k \geq 1, k|d} \frac{1}{k^2}. \end{aligned}$$

Here the second equality follows from (49) and the last equality follows from the derived equivalence (89). Therefore we obtain the desired result.  $\square$

Next we compute the invariants  $L_{n,m}$ .

**Proposition 6.8.** *We have  $L_{n,m} = 0$  for  $n \neq 0$ , and*

$$L_{0,m} = \chi(\text{Hilb}_m(S)).$$

*Here  $\text{Hilb}_m(S)$  is the Hilbert scheme of  $m$ -points in  $S$ .*

*Proof.* Let us take an ample divisor  $\omega$  on  $X$  and a stable pair

$$(90) \quad s: \mathcal{O}_X \rightarrow E,$$

with  $E$  supported on fibers of  $\pi$ . By taking the Harder-Narasimhan filtration and Jordan-Hölder filtration with respect to  $Z_\omega$ -stability, (cf. Example 2.3 (iii),) we can take a filtration of  $E$ ,

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_N = E,$$

such that each  $F_i = E_i/E_{i-1}$  is  $Z_\omega$ -stable with

$$(91) \quad \arg Z_\omega(F_i) \geq \arg Z_\omega(F_{i+1}),$$

for all  $i$ . Note that each  $F_j$  is written as  $i_{p*}F'_j$  for a stable sheaf  $F'_j$  on  $\pi^{-1}(p)$  as in (86). Also the composition,

$$\mathcal{O}_X \xrightarrow{s} E \rightarrow E/E_{N-1} = F_N,$$

should be non-zero since  $s$  is surjective in dimension one. Therefore

$$\text{Hom}_X(\mathcal{O}_X, F_N) \cong \text{Hom}_{X_p}(\mathcal{O}_{X_p}, F_N) \neq 0,$$

which implies that

$$\arg Z_\omega(F_N) \geq \arg Z_\omega(\mathcal{O}_{X_p}) = \pi/2.$$

Combined with the inequality (91), we conclude that  $\chi(E) \geq 0$ .

The above argument shows that  $P_n(X, m)$  is empty for  $n < 0$ , hence  $P_{n,m} = 0$  for  $n < 0$ . By the formula (85) and the symmetry  $L_{n,m} = L_{-n,m}$ , we conclude

$$L_{n,m} = 0, \quad \text{if } n \neq 0.$$

Let us compute  $L_{0,m}$ . By substituting  $q = 0$  in the formula (85), we have

$$(92) \quad L_{0,m} = P_{0,m}.$$

Suppose that a stable pair (90) satisfies  $\chi(E) = 0$ . Then the above argument shows that  $F_N \cong \mathcal{O}_{X_p}$ , and we obtain a morphism

$$I_p \rightarrow E_{N-1},$$

which is surjective in dimension one. Here  $I_p$  is the ideal sheaf of  $\pi^{-1}(p)$ . Repeating the above argument, we see that

$$(93) \quad F_i \cong \mathcal{O}_{X_p}, \quad \text{Cok}(s) = 0,$$

for all  $i$ . It is easy to see that a pair (90) satisfying the property (93) is obtained by the pull-back,

$$\mathcal{O}_S \twoheadrightarrow \mathcal{O}_W,$$

for a zero dimensional subscheme  $W \subset S$  of length  $m$ . Therefore we have the isomorphism,

$$P_0(X, m) \cong \text{Hilb}_m(S),$$

and

$$P_{0,m} = \chi(\text{Hilb}_m(S)).$$

Combined with (92), we obtain the desired result.  $\square$

Combining the above two proposition, we obtain the following theorem.

**Theorem 6.9.** *We have the following formula,*

$$(94) \quad \text{PT}(X/S) = \prod_{m \geq 1, j \geq 1} (1 - (-q)^j t^m)^{-j\chi(X)} (1 - t^m)^{-\chi(S)}.$$

*Proof.* By Proposition 6.7 and Theorem 6.4, the series  $\text{PT}(X/S)$  is written as a Gopakumar-Vafa form (68) with  $n_0^m$  equal to  $-\chi(X)$  for all  $m \geq 1$ . Also Proposition 6.8 implies that

$$\begin{aligned} \sum_{n,m} L_{n,m} q^n t^m &= \sum_m L_{0,m} t^m \\ &= \sum_m \chi(\text{Hilb}_m(S)) t^m \\ &= \prod_{m \geq 1} (1 - t^m)^{-\chi(S)}. \end{aligned}$$



Here the last equality is Göttsche's formula [13]. Therefore we have the desired formula.  $\square$

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